where

$$\epsilon(x,\bar{x})=+1 \quad \text{for} \quad x^4<\bar{x}^4$$
$$=-1 \quad \text{for} \quad \bar{x}^4<x^4. \tag{3.7}$$

Because of the infinite integrations in Eq. (3.6), it is clear that the additional term is either zero or infinite for a periodic motion of the system. Thus, for periodic motions the Frenkel 4-momentum either coincides with the canonical 4-momentum or gives infinite results. In our case of point charges in circular motion, the right-hand side of Eq. (3.6) vanishes, so that the Frenkel 4-momentum also leads to the energy given by Eq. (3.4).

Our system, characterized by Eqs. (3.1) to (3.5), can now be quantized by putting $L=n\hbar$. For either positronium (e electronic charge, $m=\bar{m}$ electron mass) or hydrogen (e, m electronic charge and mass, $\bar{m}$ proton mass), the resulting quantized motions are all nonrelativistic. They are the usual Bohr motions with small corrections for retardation and other relativistic effects and, in the case of hydrogen, with small corrections for the motion of the nucleus.

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Coherent and Incoherent States of the Radiation Field*

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Methods are developed for discussing the photon statistics of arbitrary radiation fields in fully quantum-mechanical terms. In order to keep the classical limit of quantum electrodynamics plainly in view, extensive use is made of the coherent states of the field. These states, which reduce the field correlation functions to factorized forms, are shown to offer a convenient basis for the description of fields of all types. Although they are not orthogonal to one another, the coherent states form a complete set. It is shown that any quantum state of the field may be expanded in terms of them in a unique way. Expansions are also developed for arbitrary operators in terms of products of the coherent state vectors. These expansions are discussed as a general method of representing the density operator for the field. A particular form is exhibited for the density operator which makes it possible to carry out many quantum-mechanical calculations by methods resembling those of classical theory. This representation permits clear insights into the essential distinction between the quantum and classical descriptions of the field. It leads, in addition, to a simple formulation of a superposition law for photon fields. Detailed discussions are given of the incoherent fields which are generated by superposing the outputs of many stationary sources. These fields are all shown to have intimately related properties, some of which have been known for the particular case of blackbody radiation.

I. INTRODUCTION

FEW problems of physics have received more attention in the past than those posed by the dual wave-particle properties of light. The story of the solution of these problems is a familiar one. It has culminated in the development of a remarkably versatile quantum theory of the electromagnetic field. Yet, for reasons which are partly mathematical and partly, perhaps, the accident of history, very little of the insight of quantum electrodynamics has been brought to bear on the problems of optics. The statistical properties of photon beams, for example, have been discussed to date almost exclusively in classical or semiclassical terms. Such discussions may indeed be informative, but they inevitably leave open serious questions of self-consistency, and risk overlooking quantum phenomena which have no classical analogs. The wave-particle duality, which should be central to any correct treatment of photon statistics, does not survive the transition to the classical limit. The need for a more consistent theory has led us to begin the development of a fully quantum-mechanical approach to the problems of photon statistics. We have quoted several of the results of this work in a recent note,[1] and shall devote much of the present paper to explaining the background of the material reported there.

Most of the mathematical development of quantum electrodynamics to date has been carried out through the use of a particular set of quantum states for the field. These are the stationary states of the noninteracting field, which corresponds to the presence of a precisely defined number of photons. The need to use these states has seemed almost axiomatic inasmuch as nearly all quantum electrodynamical calculations have been carried out by means of perturbation theory. It is characteristic of electrodynamical perturbation theory that in each successive order of approximation it describes processes which either increase or decrease the number of photons present by one. Calculations performed by such methods have only rarely been able to deal with more than a few photons at a time. The

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[1] R. J. Glauber, Phys. Rev. Letters **10**, 84 (1963).

description of the light beams which occur in optics, on the other hand, may require that we deal with states in which the number of photons present is large and intrinsically uncertain. It has long been clear that the use of the usual set of photon states as a basis offers at best only an awkward way of approaching such problems.

We have found that the use of a rather different set of states, one which arises in a natural way in the discussion of correlation and coherence[2,3] properties of fields, offers much more penetrating insights into the role played by photons in the description of light beams. These states, which we have called coherent ones, are of a type that has long been used to illustrate the time-dependent behavior of harmonic oscillators. Since they lack the convenient property of forming an orthogonal set, very little attention has been paid them as a set of basis states for the description of fields. We shall show that these states, though not orthogonal, do form a complete set and that any state of the field may be represented simply and uniquely in terms of them. By suitably extending the methods used to express arbitrary states in terms of the coherent states, we may express arbitrary operators in terms of products of the corresponding state vectors. It is particularly convenient to express the density operator for the field in an expansion of this type. Such expansions have the property that whenever the field possesses a classical limit, they render that limit evident while at the same time preserving an intrinsically quantum-mechanical description of the field.

The earlier sections of the paper are devoted to a detailed introduction of the coherent states and a survey of some of their properties. We then undertake in Secs. IV and V the expansion of arbitrary states and operators in terms of the coherent states. Section VI is devoted to a discussion of the particular properties of density operators and the way these properties are represented in the new scheme. The application of the formalism to physical problems is begun in Sec. VII, where we introduce a particular form for the density operator which seems especially suited to the treatment of radiation by macroscopic sources. This form for the density operator leads to a particularly simple way of describing the superposition of radiation fields. A form of the density operator which corresponds to a very commonly occurring form of incoherence is then discussed in Sec. VIII and shown to be closely related to the density operator for blackbody radiation. In Sec. IX the results established earlier for the treatment of single modes of the radiation field are generalized to treat the entire field. The photon fields generated by arbitrary distributions of classical currents are shown to have an especially simple description in terms of coherent states. Finally, in Sec. X the methods of the preceding sections are illustrated in a discussion of certain forms of coherent and incoherent fields and of their spectra and correlation functions.

II. FIELD-THEORETICAL BACKGROUND

We have, in an earlier paper,[3] discussed the separation of the electric field operator $\mathbf{E}(\mathbf{r}t)$ into its positive-frequency part $\mathbf{E}^{(+)}(\mathbf{r}t)$ and its negative-frequency part $\mathbf{E}^{(-)}(\mathbf{r}t)$. These individual fields were then used to define a succession of correlation functions $G^{(n)}$, the simplest of which takes the form

$$G_{\mu\nu}^{(1)}(\mathbf{r}t,\mathbf{r}'t')=\mathrm{tr}\{\rho E_\mu^{(-)}(\mathbf{r}t)E_\nu^{(+)}(\mathbf{r}'t')\}, \quad (2.1)$$

where ρ is the density operator which describes the field and the symbol tr stands for the trace. We noted, in discussing these functions, that there exist quantum-mechanical states which are eigenstates of the positive- and negative-frequency parts of the fields in the senses indicated by the relations

$$E_\mu^{(+)}(\mathbf{r}t)|\ \rangle=\mathcal{E}_\mu(\mathbf{r}t)|\ \rangle, \quad (2.2)$$

$$\langle\ |E_\mu^{(-)}(\mathbf{r}t)=\mathcal{E}_\mu^*(\mathbf{r}t)\langle\ |, \quad (2.3)$$

in which the function $\mathcal{E}_\mu(\mathbf{r}t)$ plays the role of an eigenvalue. It is possible, as we shall note, to find eigenstates $|\ \rangle$ which correspond to arbitrary choices of the eigenvalue function $\mathcal{E}_\mu(\mathbf{r}t)$, provided they obey the Maxwell equations satisfied by the field operator $E_\mu(\mathbf{r}t)$ and contain only positive frequency terms in their Fourier resolutions.

The importance of the eigenstates defined by Eqs. (2.2) and (2.3) is indicated by the fact that they cause the correlation functions to factorize. If the field is in an eigenstate of this type we have $\rho=|\ \rangle\langle\ |$, and the first-order correlation function therefore reduces to

$$G_{\mu\nu}^{(1)}(\mathbf{r}t,\mathbf{r}'t')=\mathcal{E}_\mu^*(\mathbf{r}t)\mathcal{E}_\nu(\mathbf{r}'t'). \quad (2.4)$$

An analogous separation into a product of $2n$ factors takes place in the nth- order correlation function. The existence of such factorized forms for the correlation functions is the condition we have used to define fully coherent fields. The eigenstates $|\ \rangle$, which we have therefore called the coherent states, have many properties which it will be interesting to study in detail. For this purpose, it will be useful to introduce some of the more directly related elements of quantum electrodynamics.

The electric and magnetic field operators $\mathbf{E}(\mathbf{r}t)$ and $\mathbf{B}(\mathbf{r}t)$ may be derived from the operator $\mathbf{A}(\mathbf{r}t)$, which represents the vector potential, via the relations

$$\mathbf{E}=-\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}, \quad \mathbf{B}=\nabla\times\mathbf{A}. \quad (2.5)$$

We shall find it convenient, in discussing the quantum states of the field, to describe the field by means of a discrete succession of dynamical variables rather than

[2] R. J. Glauber, in Proceedings of the Third International Conference on Quantum Electronics, Paris, France, 1963 (to be published).

[3] R. J. Glauber, Phys. Rev. **130**, 2529 (1963).

a continuum of them. For this reason we assume that the field we are discussing is confined within a spatial volume of finite size, and expand the vector potential within that volume in an appropriate set of vector mode functions. The amplitudes associated with these oscillation modes then form a discrete set of variables whose dynamical behavior is easily discussed.

The most convenient choice of a set of mode functions, $\mathbf{u}_k(\mathbf{r})$, is usually determined by physical considerations which have little direct bearing on our present work. In particular, we need not specify the nature of the boundary conditions for the volume under study; they may be either the periodic boundary conditions which lead to traveling wave modes, or the conditions appropriate to reflecting surfaces which lead to standing waves. If the volume contains no refracting materials, the mode function $\mathbf{u}_k(\mathbf{r})$, which corresponds to frequency ω_k, may be taken to satisfy the wave equation

$$\nabla^2\mathbf{u}_k+\frac{\omega_k^2}{c^2}\mathbf{u}_k=0 \tag{2.6}$$

at interior points. More generally, whatever the form of the wave equation or the boundary conditions may be, we shall assume that the mode functions form a complete set which satisfies the orthonormality condition

$$\int\mathbf{u}_k^*(\mathbf{r})\cdot\mathbf{u}_l(\mathbf{r})d\mathbf{r}=\delta_{kl}, \tag{2.7}$$

and the transversality condition

$$\nabla\cdot\mathbf{u}_k(\mathbf{r})=0. \tag{2.8}$$

The plane-wave mode functions appropriate to a cubical volume of side L may be written as

$$\mathbf{u}_k(\mathbf{r})=L^{-3/2}\hat{e}^{(\lambda)}\exp(i\mathbf{k}\cdot\mathbf{r}), \tag{2.9}$$

where $\hat{e}^{(\lambda)}$ is a unit polarization vector. This example illustrates the way in which the mode index k may represent an abbreviation for several discrete variables, i.e., in this case the polarization index ($\lambda=1,2$) and the three Cartesian components of the propagation vector $\mathbf{k}$. The polarization vector $\hat{e}^{(\lambda)}$ is required to be perpendicular to $\mathbf{k}$ by the condition (2.8), and the permissible values of $\mathbf{k}$ are determined in a familiar way by means of periodic boundary conditions.

The expansion we shall use for the vector potential takes the form

$$\mathbf{A}(\mathbf{r}t)=c\sum_k\left(\frac{\hbar}{2\omega_k}\right)^{1/2}\times(a_k\mathbf{u}_k(\mathbf{r})e^{-i\omega_k t}+a_k^\dagger\mathbf{u}_k^*(\mathbf{r})e^{i\omega_k t}), \tag{2.10}$$

in which the normalization factors have been chosen to render dimensionless the pair of complex-conjugate amplitudes a_k and $a_k^\dagger$. In the classical form of electromagnetic theory these Fourier amplitudes are complex numbers which may be chosen arbitrarily but remain constant in time when no charges or currents are present. In quantum electrodynamics, on the other hand, these amplitudes must be regarded as mutually adjoint operators. The amplitude operators, as we have defined them, will likewise remain constant when no field sources are active in the system studied.

The dynamical behavior of the field amplitudes is governed by the electromagnetic Hamiltonian which, in rationalized units, takes the form

$$H=\tfrac{1}{2}\int(\mathbf{E}^2+\mathbf{B}^2)d\mathbf{r}. \tag{2.11}$$

With the use of Eqs. (2.7,8) and of a suitable set of boundary conditions on the mode functions, the Hamiltonian may be reduced to the form

$$H=\tfrac{1}{2}\sum_k\hbar\omega_k(a_k^\dagger a_k+a_k a_k^\dagger). \tag{2.12}$$

This expression is the source of a well-known and extremely fruitful analogy between the mode amplitudes of the field and the coordinates of an assembly of one-dimensional harmonic oscillators. The quantum mechanical properties of the amplitude operators a_k and $a_k^\dagger$ may be described completely by adopting for them the commutation relations familiar from the example of independent harmonic oscillators:

$$[a_k,a_{k'}]=[a_k^\dagger,a_{k'}^\dagger]=0, \tag{2.13a}$$

$$[a_k,a_{k'}^\dagger]=\delta_{kk'}. \tag{2.13b}$$

Having thus separated the dynamical variables of the different modes, we are now free to discuss the quantum states of the modes independently of one another. Our knowledge of the state of each mode may be described by a state vector $|\ \rangle_k$ in a Hilbert space appropriate to that mode. The states of the entire field are then defined in the product space of the Hilbert spaces for all of the modes.

To discuss the quantum states of the individual modes we need only be familiar with the most elementary aspects of the treatment of a single harmonic oscillator. The Hamiltonian $\frac{1}{2}h\omega_k(a_k^\dagger a_k+a_k a_k^\dagger)$ has eigenvalues $h\omega_k(n_k+\frac{1}{2})$, where n_k is an integer ($n_k=0,1,2\cdots$). The state vector for the ground state of the oscillator will be written as $|\ \rangle_k$. It is defined by the condition

$$a_k|0\rangle_k=0. \tag{2.14}$$

The state vectors for the excited states of the oscillator may be obtained by applying integral powers of the operator $a_k^\dagger$ to $|0\rangle_k$. These states are written in normalized form as

$$|n_k\rangle_k=\frac{(a_k^\dagger)^{n_k}}{(n_k!)^{1/2}}|0\rangle_k, \quad (n_k=0,1,2\cdots). \tag{2.15}$$

The way in which the operators a_k and $a_k^\dagger$ act upon these states is indicated by the relations

$$a_k|n_k\rangle_k = n_k^{1/2}|n_k-1\rangle_k, \tag{2.16}$$

$$a_k^\dagger|n_k\rangle = (n_k+1)^{1/2}|n_k+1\rangle_k, \tag{2.17}$$

$$a_k^\dagger a_k|n_k\rangle = n_k|n_k\rangle. \tag{2.18}$$

With these preliminaries completed we are now ready to discuss the coherent states of the field in greater detail. The expansion (2.10) for the vector potential exhibits its positive frequency part as the sum containing the photon annihilation operators a_k and its negative frequency part as that involving the creation operators $a_k^\dagger$. The positive frequency part of the electric field operator is thus given, according to (2.10), by

$$\mathbf{E}^{(+)}(\mathbf{r}t) = i\sum_k (\tfrac{1}{2}\hbar\omega_k)^{1/2} a_k \mathbf{u}_k(\mathbf{r}) e^{-i\omega_k t}. \tag{2.19}$$

The eigenvalue functions $\mathcal{E}(\mathbf{r}t)$ defined by Eq. (2.2) must clearly satisfy the Maxwell equations, just as the operator $\mathbf{E}^{(+)}(\mathbf{r}t)$ does. They therefore possess an expansion in normal modes similar to Eq. (2.19). In other words we may introduce a set of c-number Fourier coefficients α_k which permit us to write the eigenvalue function as

$$\mathcal{E}(\mathbf{r}t) = i\sum_k (\tfrac{1}{2}\hbar\omega_k)^{1/2} \alpha_k \mathbf{u}_k(\mathbf{r}) e^{-i\omega_k t}. \tag{2.20}$$

Since the mode functions $\mathbf{u}_k(\mathbf{r})$ form an orthogonal set, it then follows that the eigenstate $|\ \rangle$ for the field obeys the infinite succession of relations

$$a_k|\ \rangle = \alpha_k|\ \rangle, \tag{2.21}$$

for all modes k. To find the states which satisfy these relations we seek states, $|\alpha_k\rangle_k$, of the individual modes which individually obey the relations

$$a_k|\alpha_k\rangle_k = \alpha_k|\alpha_k\rangle_k. \tag{2.22}$$

The coherent states $|\ \rangle$ of the field, considered as a whole, are then seen to be direct products of the individual states $|\alpha_k\rangle$,

$$|\ \rangle = \prod_k |\alpha_k\rangle_k. \tag{2.23}$$

III. COHERENT STATES OF A SINGLE MODE

The next few sections will be devoted to discussing the description of a single mode oscillator. We may therefore simplify the notation a bit by dropping the mode index k as a subscript to the state vector and to the amplitude parameters and operators. To find the oscillator state $|\alpha\rangle$ which satisfies

$$a|\alpha\rangle = \alpha|\alpha\rangle, \tag{3.1}$$

we begin by taking the scalar product of both sides of the equation with the nth excited state, $\langle n|$. By using the Hermitian adjoint form of the relation (2.17), we find the recursion relation

$$(n+1)^{1/2}\langle n+1|\alpha\rangle = \alpha\langle n|\alpha\rangle \tag{3.2}$$

for the scalar products $\langle n|\alpha\rangle$. We immediately find from the recursion relation that

$$\langle n|\alpha\rangle = \frac{\alpha^n}{(n!)^{1/2}}\langle 0|\alpha\rangle. \tag{3.3}$$

These scalar products are the expansion coefficients of the state $|\alpha\rangle$ in terms of the complete orthonormal set $|n\rangle$ $(n=0, 1, \cdots)$. We thus have

$$|\alpha\rangle = \sum_n |n\rangle\langle n|\alpha\rangle = \langle 0|\alpha\rangle \sum_n \frac{\alpha^n}{(n!)^{1/2}}|n\rangle. \tag{3.4}$$

The squared length of the vector $|\alpha\rangle$ is thus

$$\langle\alpha|\alpha\rangle = |\langle 0|\alpha\rangle|^2 \sum_n \frac{|\alpha|^{2n}}{n!} = |\langle 0|\alpha\rangle|^2 e^{|\alpha|^2}. \tag{3.5}$$

If the state $|\alpha\rangle$ is normalized so that $\langle\alpha|\alpha\rangle = 1$ we may evidently define its phase by choosing

$$\langle 0|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2}. \tag{3.6}$$

The coherent states of the oscillator therefore take the forms

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_n \frac{\alpha^n}{(n!)^{1/2}}|n\rangle \tag{3.7}$$

and

$$\langle\alpha| = e^{-\frac{1}{2}|\alpha|^2} \sum_n \frac{(\alpha^*)^n}{(n!)^{1/2}}\langle n|. \tag{3.8}$$

These forms show that the average occupation number of the nth state is given by a Poisson distribution with mean value $|\alpha|^2$,

$$|\langle n|\alpha\rangle|^2 = \frac{|\alpha|^{2n}}{n!}e^{-|\alpha|^2}. \tag{3.9}$$

They also show that the coherent state $|\alpha\rangle$ corresponding to $\alpha=0$ is the unique ground state of the oscillator, i.e., the state $|n\rangle$ for $n=0$.

An alternative approach to the coherent states will also prove quite useful in the work to follow. For this purpose we assume that there exists a unitary operator D which acts as a displacement operator upon the amplitudes $a^\dagger$ and a. We let D be a function of a complex parameter β, and require that it displace the amplitude operators according to the scheme

$$D^{-1}(\beta)aD(\beta) = a+\beta, \tag{3.10}$$

$$D^{-1}(\beta)a^\dagger D(\beta) = a^\dagger+\beta^*. \tag{3.11}$$

Then if $|\alpha\rangle$ obeys Eq. (3.1), it follows that $D^{-1}(\beta)|\alpha\rangle$ is an eigenstate of a corresponding to the eigenvalue $\alpha-\beta$,

$$aD^{-1}(\beta)|\alpha\rangle=(\alpha-\beta)D^{-1}(\beta)|\alpha\rangle. \tag{3.12}$$

In particular, if we choose $\beta=\alpha$, we find

$$aD^{-1}(\alpha)|\alpha\rangle=0.$$

Since the ground state of the oscillator is uniquely defined by the relation (2.14), it follows that $D^{-1}(\alpha)|\alpha\rangle$ is just the ground state, $|0\rangle$. The coherent states, in other words, are just displaced forms of the ground state of the oscillator,

$$|\alpha\rangle=D(\alpha)|0\rangle. \tag{3.13}$$

To find an explicit form for the displacement operator $D(\alpha)$, we begin by considering infinitesimal displacements in the neighborhood of $D(0)=1$. For arbitrary displacements $d\alpha$, we see easily from the commutation rules (2.13) that $D(d\alpha)$ may be chosen to have the form

$$D(d\alpha)=1+a^\dagger d\alpha-ad\alpha^*, \tag{3.14}$$

which holds to first order in $d\alpha$. To formulate a simple differential equation obeyed by the unknown operator we consider increments of α of the form $d\alpha=\alpha d\lambda$ where λ is a real parameter. Then if we assume the operators D to possess the group multiplication property

$$D(\alpha(\lambda+d\lambda))=D(\alpha d\lambda)D(\alpha\lambda), \tag{3.15}$$

we find the differential equation

$$\frac{d}{d\lambda}D(\alpha\lambda)=(\alpha a^\dagger-\alpha^* a)D(\alpha\lambda), \tag{3.16}$$

whose solution, evaluated for $\lambda=1$, is the unitary operator

$$D(\alpha)=e^{\alpha a^\dagger-\alpha^* a}. \tag{3.17}$$

The coherent states $|\alpha\rangle$ may therefore be written in the form

$$|\alpha\rangle=e^{\alpha a^\dagger-\alpha^* a}|0\rangle \tag{3.18}$$

which is correctly normalized since $D(\alpha)$ is unitary.

It is interesting to discuss the relationship between the two forms we have derived for the coherent states. For this purpose we invoke a simple theorem on the multiplication of exponential functions of operators. If $\mathfrak{A}$ and $\mathfrak{B}$ are any two operators, whose commutator $[\mathfrak{A},\mathfrak{B}]$ commutes with each of them,

$$[[\mathfrak{A},\mathfrak{B}],\mathfrak{A}]=[[\mathfrak{A},\mathfrak{B}],\mathfrak{B}]=0, \tag{3.19}$$

it may be shown[4] that

$$\exp(\mathfrak{A})\exp(\mathfrak{B})=\exp\{\mathfrak{A}+\mathfrak{B}+\tfrac{1}{2}[\mathfrak{A},\mathfrak{B}]\}. \tag{3.20}$$

If we write $\mathfrak{A}=a^\dagger$ and $\mathfrak{B}=a$, this theorem permits us to resolve the exponential $D(\alpha)$ given by Eq. (3.17) into the product

$$D(\alpha)=e^{-\frac{1}{2}|\alpha|^2}e^{\alpha a^\dagger}e^{-\alpha^* a}. \tag{3.21}$$

Products of this type, which have been ordered so that the annihilation operators all stand to the right of the creation operators, will be said to be in normal form. Their convenience is indicated by the fact that the exponential $\exp[-\alpha^* a]$, when applied to the ground state $|0\rangle$, reduces in effect to unity, i.e., we have

$$e^{-\alpha^* a}|0\rangle=|0\rangle, \tag{3.22}$$

since the exponential may be expanded in series and the definition (2.14) of the ground state applied. It follows then that the coherent states may be written as

$$|\alpha\rangle=D(\alpha)|0$$

$$=e^{-\frac{1}{2}|\alpha|^2}e^{\alpha a^\dagger}|0\rangle \tag{3.23}$$

$$=e^{-\frac{1}{2}|\alpha|^2}\sum_n\frac{(\alpha a^\dagger)^n}{n!}|0\rangle. \tag{3.24}$$

Since the excited states of the oscillator are given by $|n\rangle=(n!)^{-1/2}(a^\dagger)^n|0\rangle$, we have once again derived the expression

$$|\alpha\rangle=e^{-\frac{1}{2}|\alpha|^2}\sum_n\frac{\alpha^n}{n!}|n\rangle.$$

It may help in visualizing the coherent states if we discuss the form they take in coordinate space and in momentum space. We therefore introduce a pair of Hermitian operators q and p to represent, respectively, the coordinate of the mode oscillator and its momentum. These operators, which must satisfy the canonical commutation relation, $[q,p]=i\hbar$, may be defined for our purposes by the familiar expressions

$$q=(\hbar/2\omega)^{1/2}(a^\dagger+a), \tag{3.25a}$$

$$p=i(\hbar\omega/2)^{1/2}(a^\dagger-a). \tag{3.25b}$$

To find the expectation value of q and p in the coherent states we need only use Eq. (3.1), which defines these states, and its corresponding Hermitian adjoint form. We have then

$$\langle\alpha|q|\alpha\rangle=(2\hbar/\omega)^{1/2}\,\mathrm{Re}\,\alpha, \tag{3.26a}$$

$$\langle\alpha|p|\alpha\rangle=(2\hbar\omega)^{1/2}\,\mathrm{Im}\,\alpha, \tag{3.26b}$$

where $\mathrm{Re}\,\alpha$ and $\mathrm{Im}\,\alpha$ stand for the real and imaginary parts of α.

To find the wave functions for the coherent states, we write the defining equation (3.1) in the form

$$(2\hbar\omega)^{-1/2}(\omega q+ip)|\alpha\rangle=\alpha|\alpha\rangle, \tag{3.27}$$

and take the scalar product of both members with the conjugate state $\langle q'|$, which corresponds to the eigenvalue q' for q. Since the momentum may be represented by a derivative operator, i.e., $\langle q'|p=-i\hbar(d/dq')\langle q'|$, we find that the coordinate space wave function, $\langle q'|\alpha\rangle$,

[4] A. Messiah, *Quantum Mechanics* (North-Holland Publishing Company, Amsterdam, 1961), Vol. I, p. 442.

obeys the differential equation

$$\frac{d}{dq'}\langle q'|\alpha\rangle=-2\left(\frac{\omega}{2\hbar}\right)^{1/2}\left\{\left(\frac{\omega}{2\hbar}\right)^{1/2}q'-\alpha\right\}\langle q'|\alpha\rangle. \quad (3.28)$$

The equation may be integrated immediately to yield a solution for the wave function which, in normalized form, is

$$\langle q'|\alpha\rangle=(\omega/\pi\hbar)^{1/4}\exp\{-[(\omega/2\hbar)^{1/2}q'-\alpha]^2\}. \quad (3.29)$$

An analogous argument furnishes the momentum space wave function. If we take the scalar product of Eq. (3.27) with a momentum eigenstate $\langle p'|$, and use the relation $\langle p'|q=i\hbar(\partial/\partial p')\langle p'|$, we reach a differential equation whose normalized solution is

$$\langle p'|\alpha\rangle=(\pi\hbar\omega)^{-1/4}\exp\{-[(2\hbar\omega)^{-1/2}p'+i\alpha]^2\}. \quad (3.30)$$

Both of these wave functions are simply displaced forms of the ground-state wave function of the oscillator. The parameters $(\hbar/\omega)^{1/2}$ and $(\hbar\omega)^{1/2}$ correspond to the amplitudes of the zero-point fluctuations of the coordinate and momentum, respectively, for an oscillator of unit mass. The fact that the wave functions for the coherent states have this elementary structure should be no surprise in view of the way they are generated in Eq. (3.13), by means of displacements in the complex α plane.

The time-independent states $|\alpha\rangle$ which we have been describing are those characteristic of the Heisenberg picture of quantum mechanics. The Schrödinger picture, alternatively, would make use of the time-dependent states $\exp(-iHt/\hbar)|\alpha\rangle$. If we omit the zero-point energy $\frac{1}{2}\hbar\omega$ from the oscillator Hamiltonian and write $H=\hbar\omega a^\dagger a$, it is then clear from the expansion (3.7) for $|\alpha\rangle$ that the corresponding Schrödinger state takes the same form with α replaced by $\alpha e^{-i\omega t}$. We may thus write the Schrödinger state as $|ae^{-i\omega t}\rangle$. With the substitution of $\alpha e^{-i\omega t}$ for α in Eqs. (3.26a) and (3.26b), we see that the expectation values of the coordinate and momentum carry out a simple harmonic motion with coordinate amplitude $(2\hbar/\omega)^{1/2}|\alpha|$. The same substitutions in the wave functions (3.29) and (3.30) show that the Gaussian probability densities characteristic of the ground state of the oscillator are simply carried back and forth in the same motion as the expectation values. Such wave packets are, of course, quite familiar; they were introduced to quantum mechanics at a very early stage by Schrödinger,[5] and have often been used to illustrate the way in which the behavior of the oscillator approaches the classical limit.

Another connection in which the wave packets (3.29) and (3.30) have been discussed in the past has to do with the particular way in which they localize the coordinate q' and the momentum p'. Wave packets can, of course, be found which localize either variable more sharply, but only at the expense of the localization of the other. There is a sense in which the wave packets (3.29) and (3.30) furnish a unique compromise; they minimize the product of the uncertainties of the variables q' and p'. If we represent expectation values by means of the angular brackets $\langle\ \rangle$ and define the variances

$$(\Delta q)^2=\langle q^2\rangle-\langle q\rangle^2, \quad (3.31a)$$

$$(\Delta p)^2=\langle p^2\rangle-\langle p\rangle^2, \quad (3.31b)$$

we find, for the wave functions (3.29) and (3.30), that the product of the variances is

$$(\Delta p)^2(\Delta q)^2=\tfrac{1}{4}\hbar^2.$$

According to the uncertainty principle, this is the minimum value such a product can have.[6] There thus exists a particular sense in which the description of an oscillator by means of the wave functions (3.29) and (3.30) represents as close an approach to classical localization as is possible.

The uses we shall make of the coherent states in quantum electrodynamics will not, in fact, require the explicit introduction of coordinate or momentum variables. We have reviewed the familiar representations of the coherent states in terms of these variables in the hope that they may be of some help in understanding the various applications of the states which we shall shortly undertake.

One property of the states $|\alpha\rangle$ which is made clear by the wave-function representations is that two such states are not, in general, orthogonal to one another. If we consider, for example, the wave functions $\langle q'|\alpha\rangle$ and $\langle q'|\alpha'\rangle$ for values of α' close to α, it is evident that the functions are similar in form and overlap one another appreciably. For values of α' quite different from α, however, the overlap is at most quite small. We may therefore expect that the scalar product $\langle\alpha|\alpha'\rangle$, which is unity for $\alpha'=\alpha$, will tend to decrease in absolute magnitude as α' and α recede from one another in the complex plane. The scalar product may, in fact, be calculated more simply than by using wave functions if we employ the representations (3.7) and (3.8). We then find

$$\langle\alpha|\beta\rangle=e^{-\frac{1}{2}|\alpha|^2-\frac{1}{2}|\beta|^2}\sum_{n,m}\frac{(\alpha^*)^n\beta^m}{(n!m!)^{1/2}}\langle n|m\rangle,$$

which, in view of the orthonormality of the $|n\rangle$ states, reduces to

$$\langle\alpha|\beta\rangle=\exp\{\alpha^*\beta-\tfrac{1}{2}|\alpha|^2-\tfrac{1}{2}|\beta|^2\}. \quad (3.32)$$

The absolute magnitude of the scalar product is given by

$$|\langle\alpha|\beta\rangle|^2=\exp\{-|\alpha-\beta|^2\}, \quad (3.33)$$

[5] E. Schrödinger, Naturwissenschaften **14**, 664 (1926). For a more recent treatment see L. I. Schiff, *Quantum Mechanics* (McGraw-Hill Book Company, Inc., New York, 1955), 2nd ed., p. 67.

[6] W. Heisenberg, *The Physical Principles of the Quantum Theory* (University of Chicago Press, Chicago, 1930, reprinted by Dover Publications, Inc., New York, 1930), pp. 16–19.

which shows that the coherent states tend to become approximately orthogonal for values of α and β which are sufficiently different. The fact that these states are not even approximately orthogonal for $|\alpha-\beta|$ of order unity may be regarded as an expression of the overlap caused by the presence of the displaced zero-point fluctuations.

Since the coherent states do not form an orthogonal set, they appear to have received little attention as a possible system of basis vectors for the expansion of arbitrary states.[7] We shall show in the following section that such expansions can be carried out conveniently and uniquely and that they possess exceedingly useful properties. In later sections we shall, by generalizing the procedure to deal with bilinear combinations of states $|\alpha\rangle$ and $\langle\beta|$, develop analogous expansions for operators[1] as well.

IV. EXPANSION OF ARBITRARY STATES IN TERMS OF COHERENT STATES

While orthogonality is a convenient property for a set of basis states it is not a necessary one. The essential property of such a set is that it be complete. The set of coherent states $|\alpha\rangle$ for a mode oscillator can be shown without difficulty to form a complete set. To give a proof we need only demonstrate that the unit operator may be expressed as a suitable sum or an integral, over the complex α plane, of projection operators of the form $|\alpha\rangle\langle\alpha|$. In order to describe such integrals we introduce the differential element of area in the α plane

$$d^2\alpha = d(\mathrm{Re}\,\alpha)d(\mathrm{Im}\,\alpha) \tag{4.1}$$

(i.e., $d^2\alpha$ is real). If we write $\alpha=|\alpha|e^{i\vartheta}$, we may easily prove the integral identity

$$\begin{aligned}\int (\alpha^*)^n\alpha^m e^{-|\alpha|^2}d^2\alpha &= \int_0^\infty |\alpha|^{n+m+1}e^{-|\alpha|^2}d|\alpha| \int_0^{2\pi} e^{i(m-n)\vartheta}d\vartheta \\ &= \pi n!\delta_{nm}, \end{aligned}\tag{4.2}$$

in which the integration is carried out, as indicated, over the entire area of the complex plane. With the aid of this identity and the expansions (3.7,8) for the coherent states, we may immediately show

$$\int |\alpha\rangle\langle\alpha|\,d^2\alpha = \pi\sum_n |n\rangle\langle n|\,.$$

Since the n-quantum states are known to form a complete orthonormal set, the indicated sum over n is simply the unit operator. We have thus shown[1]

$$\frac{1}{\pi}\int |\alpha\rangle\langle\alpha|\,d^2\alpha = 1\,, \tag{4.3}$$

which is a completeness relation for the coherent states of precisely the type desired.

An arbitrary state of an oscillator must possess an expansion in terms of the n-quantum states of the form

$$\begin{aligned}|\,\rangle &= \sum_n c_n|n\rangle\,, \\ &= \sum_n c_n\frac{(a^\dagger)^n}{(n!)^{1/2}}|0\rangle\,, \end{aligned}\tag{4.4}$$

where $\sum |c_n|^2=1$. The series which occurs in Eq. (4.4) may be used to define a function f of a complex variable z,

$$f(z)=\sum c_n\frac{z^n}{(n!)^{1/2}}\,. \tag{4.5}$$

It is clear from the normalization condition on the c_n that this series converges for all finite z, and thus represents a function which is analytic throughout the finite complex plane. We shall speak of the functions $f(z)$ for which $\sum|c_n|^2=1$ as the set of normalized entire functions. There is evidently a one-to-one correspondence which exists between such entire functions and the states of the oscillator. One way of approaching the description of the oscillator is to regard the functions $f(z)$ themselves as the elements of a Hilbert space. The properties of this space and of expansions carried out in it have been studied in some detail by Segal[8] and Bargmann.[9] The method we shall use for expanding arbitrary states in terms of the coherent states has been developed as a simple generalization of the usual method for carrying out changes of basis states in quantum mechanics. It is evidently equivalent, however, to one of the expansions stated by Bargmann.

If we designate the arbitrary state which corresponds to the function $f(z)$ by $|f\rangle$, then we may rewrite Eq. (4.4) as

$$|f\rangle = f(a^\dagger)|0\rangle. \tag{4.6}$$

To secure the expansion of $|f\rangle$ in terms of the states $|\alpha\rangle$, we multiply $|f\rangle$ by the representation (4.3) of the unit operator. We then find

$$|f\rangle = \frac{1}{\pi}\int |\alpha\rangle\langle\alpha|\,f(a^\dagger)|0\rangle d^2\alpha\,,$$

[7] Uses of these states as generating functions for the n-quantum states have, however, been made by J. Schwinger, Phys. Rev. **91**, 728 (1953).

[8] I. E. Segal, Illinois J. Math. **6**, 520 (1962).

[9] V. Bargmann, Commun. Pure and Appl. Math. **14**, 187 (1961); Proc. Natl. Acad. Sci. U. S. **48**, 199 (1962).

which reduces, since $\langle\alpha|f(\alpha^\dagger)=\langle\alpha|f(\alpha^*)$, to

$$|f\rangle=\frac{1}{\pi}\int|\alpha\rangle f(\alpha^*)e^{-\frac{1}{2}|\alpha|^2}d^2\alpha, \tag{4.7}$$

which is an expansion of the desired type.

It is worth noting that the expansion (4.7) can easily be inverted to furnish an explicit form for the function $f(\alpha^*)$ which corresponds to any vector $|f\rangle$. For this purpose we take the scalar product of both sides of Eq. (4.7) with the coherent state $\langle\beta|$, and then, using Eq. (3.32), evaluate the scalar product $\langle\beta|\alpha\rangle$ to find

$$\langle\beta|f\rangle=\frac{1}{\pi}e^{-\frac{1}{2}|\beta|^2}\int e^{\beta^*\alpha-|\alpha|^2}f(\alpha^*)d^2\alpha. \tag{4.8}$$

Since $f(\alpha^*)$ may be expanded in a convergent power series we note the relation

$$\frac{1}{\pi}\int e^{\beta^*\alpha-|\alpha|^2}(\alpha^*)^n d^2\alpha=(\beta^*)^n, \tag{4.9}$$

from which we may derive the more general identity

$$\frac{1}{\pi}\int e^{\beta^*\alpha-|\alpha|^2}f(\alpha^*)d^2\alpha=f(\beta^*). \tag{4.10}$$

On substituting the latter identity in Eq. (4.8) we find

$$f(\beta^*)=e^{\frac{1}{2}|\beta|^2}\langle\beta|f\rangle. \tag{4.11}$$

There is thus a unique correspondence between functions $f(\alpha^*)$ which play the role of expansion amplitudes in Eq. (4.7) and the vectors $|f\rangle$ which describe the state of the oscillator.

An expansion analogous to Eq. (4.7) also exists for the adjoint state vectors. If we let $g(\alpha^*)$ be an entire function of α^* we may construct for the state $\langle g|$ the expansion

$$\langle g|=\frac{1}{\pi}\int[g(\beta^*)]^*\langle\beta|e^{-\frac{1}{2}|\beta|^2}d^2\beta. \tag{4.12}$$

The scalar product of the two states $\langle g|$ and $|f\rangle$ may then be expressed as

$$\langle g|f\rangle=\pi^{-2}\int[g(\beta^*)]^*f(\alpha^*)\exp\{\beta^*\alpha-|\alpha|^2-|\beta|^2\}d^2\alpha d^2\beta.$$

The identity (4.10) permits us to carry out the integration over the variable α to find

$$\langle g|f\rangle=\frac{1}{\pi}\int[g(\beta^*)]^*f(\beta^*)e^{-|\beta|^2}d^2\beta. \tag{4.13}$$

This expression for the scalar product of two vectors is, in essence, the starting point used by Bargmann in his discussion[10] of the Hilbert space of functions $f(z)$.

[10] Some of Bargmann's arguments are summarized by S. Schweber, J. Math. Phys. **3**, 831 (1962), who has used them in connection with the formulation of quantum mechanics in terms of Feynman amplitudes. We are indebted to Dr. S. Bergmann for calling this reference to our attention.

It may be worth noting, for its mathematical interest, that the coherent states $|\alpha\rangle$ are not linearly independent of one another, as the members of a complete orthogonal set would be. Thus, for example, the expansion (4.7) may be used to express any given coherent state linearly in terms of all of the others, i.e., in view of Eqs. (4.11) and (3.32) we may write

$$|\alpha\rangle=\frac{1}{\pi}\int|\beta\rangle e^{\beta^*\alpha-\frac{1}{2}|\alpha|^2-\frac{1}{2}|\beta|^2}d^2\beta. \tag{4.14}$$

There exist many other types of linear dependence among the states $|\alpha\rangle$. We may, for example, note the identity

$$\int|\alpha\rangle\alpha^n e^{-\frac{1}{2}|\alpha|^2}d^2\alpha=0, \tag{4.15}$$

which holds for all integral $n>0$. It is clear from the latter result that if we admitted as expansion coefficients in Eq. (4.7) more general functions than $f(\alpha^*)$, say functions $F(\alpha,\alpha^*)$, there would be many additional ways of expanding any state in terms of coherent states. The constraint implicit in Eq. (4.7), that the expansion function must depend analytically upon the variable α^* is what renders the expansion unique. The virtue of an expansion scheme in which the coefficients are uniquely determined is evident. It becomes possible, by inverting the expansion as in Eq. (4.11), to construct an explicit solution for the expansion coefficient of any state, no matter what representation it was expressed in initially.

V. EXPANSION OF OPERATORS IN TERMS OF COHERENT STATE VECTORS

Our knowledge of the condition of an oscillator mode is rarely explicit enough in practice to permit the specification of its quantum state. Instead, we must describe it in terms of a mixture of states which is expressed by means of a density operator. The same reasons that lead us to express arbitrary states in terms of the coherent states, therefore, suggest that we develop an expansion for the density operator in terms of these states as well. We shall begin by considering in the present section a rather more general class of operators and then specialize to the case of the density operator in the section which follows.

A general quantum mechanical operator T may be expressed in terms of its matrix elements connecting states with fixed numbers of quanta as

$$T=\sum_{n,m}|n\rangle T_{nm}\langle m|, \tag{5.1}$$

$$=\sum T_{nm}(n!m!)^{-1/2}(a^\dagger)^n|0\rangle\langle 0|a^m. \tag{5.2}$$

If we use this expression for T to calculate the matrix element which connects the two coherent states $\langle\alpha|$ and $\langle\beta|$ we find

$$\langle\alpha|T|\beta\rangle=\sum_{n,m}T_{nm}(n!m!)^{-1/2}(\alpha^*)^n\beta^m\langle\alpha|0\rangle\langle 0|\beta\rangle. \quad (5.3)$$

It is evidently convenient to define a function $\mathcal{T}(\alpha^*,\beta)$ as

$$\mathcal{T}(\alpha^*,\beta)=\sum_{n,m}T_{nm}(n!m!)^{-1/2}(\alpha^*)^n\beta^m. \quad (5.4)$$

The operators which occur in quantum mechanics are often unbounded ones such as those of Eqs. (2.16)–(2.18). Those operators and the others we are apt to encounter have the property that the magnitudes of the matrix elements T_{nm} are dominated by an expression of the form Mn^jm^k for some fixed positive values of M, j, and k. It then follows that the double series (5.4) converges throughout the finite α^* and β planes and represents an entire function of both variables.

To secure the expansion of the operator T in terms of the coherent states, we may use the representation (4.3) of the unit operator to write

$$T=\frac{1}{\pi^2}\int|\alpha\rangle\langle\alpha|T|\beta\rangle\langle\beta|d^2\alpha d^2\beta, \quad (5.5)$$

$$=\frac{1}{\pi^2}\int|\alpha\rangle\mathcal{T}(\alpha^*,\beta)\langle\beta|\alpha|0\rangle\langle 0|\beta\rangle d^2\alpha d^2\beta,$$

$$=\frac{1}{\pi^2}\int|\alpha\rangle\mathcal{T}(\alpha^*,\beta)\langle\beta|\exp\{-\tfrac{1}{2}|\alpha|^2-\tfrac{1}{2}|\beta|^2\}d^2\alpha d^2\beta. \quad (5.6)$$

The inversion of this expansion, or the solution for $\mathcal{T}(\alpha^*,\beta)$, is accomplished by the same method we used to invert Eq. (4.7) and secure the amplitude function (4.11). The result of the inversion is

$$\mathcal{T}(\alpha^*,\beta)=\langle\alpha|T|\beta\rangle\exp\{\tfrac{1}{2}|\alpha|^2+\tfrac{1}{2}|\beta|^2\}. \quad (5.7)$$

We see, thus, that the expansion of operators, as well as of arbitrary quantum states, in terms of the coherent states is a unique one.

The law of operator multiplication is easily expressed in terms of the functions $\mathcal{T}$. If $T=T_1T_2$ and $\mathcal{T}_1$ and $\mathcal{T}_2$ are the functions appropriate to the latter two operators, we note that

$$\langle\alpha|T|\beta\rangle=\langle\alpha|T_1T_2|\beta\rangle$$

$$=\frac{1}{\pi}\int\langle\alpha|T_1|\gamma\rangle\langle\gamma|T_2|\beta\rangle d^2\gamma. \quad (5.8)$$

The function $\mathcal{T}$ which represents the product is therefore given by

$$\mathcal{T}(\alpha^*,\beta)=\frac{1}{\pi}\int\mathcal{T}_1(\alpha^*,\gamma)\mathcal{T}_2(\gamma^*,\beta)e^{-|\gamma|^2}d^2\gamma. \quad (5.9)$$

The expansion function for the operator $T^\dagger$, the Hermitian adjoint of T, is obtained by substituting $T_{mn}{}^*$ for T_{nm} in Eq. (5.4). It is given by $[\mathcal{T}(\beta^*,\alpha)]^*$. If the operator T is Hermitian the function $\mathcal{T}$ must satisfy the identity

$$\mathcal{T}(\alpha^*,\beta)=[\mathcal{T}(\beta^*,\alpha)]^*, \quad (5.10)$$

since the expansions of T and $T^\dagger$ are unique.

The functions $\mathcal{T}(\alpha^*,\beta)$ which represent normal products of the operators $a^\dagger$ and a such as $(a^\dagger)^n a^m$ are immediately seen from Eqs. (5.7) and (3.32) to be

$$\mathcal{T}(\alpha^*,\beta)=(\alpha^*)^n\beta^m\exp[\alpha^*\beta]. \quad (5.11)$$

In particular, the unit operator corresponds to $n=m=0$.

It may be worth noting at this point that many of the foregoing formulas can be abbreviated somewhat by adopting a normalization different from the conventional one for the coherent states. If we introduce the symbol $\|\alpha\rangle$ for the states normalized in the new way and define these as

$$\|\alpha\rangle=|\alpha\rangle e^{\frac{1}{2}|\alpha|^2}, \quad (5.12)$$

then we may write the scalar product of two such states as $\langle\alpha\|\beta\rangle$. We see from Eq. (3.32) that this scalar product is

$$\langle\alpha\|\beta\rangle=\exp[\alpha^*\beta]. \quad (5.13)$$

We may next, following Bargmann,[9] introduce an element of measure $d\mu(\alpha)$ which is defined as

$$d\mu(\alpha)=\frac{1}{\pi}e^{-|\alpha|^2}d^2\alpha. \quad (5.14)$$

With these alterations, all of the Gaussian functions, and factors of π, in the preceding formulas become absorbed, as it were, into the notation. The Eqs. (5.6) and (5.7), for example, reduce to the briefer forms

$$T=\int\|\alpha\rangle\mathcal{T}(\alpha^*,\beta)\langle\beta\|d\mu(\alpha)d\mu(\beta) \quad (5.15)$$

and

$$\mathcal{T}(\alpha^*,\beta)=\langle\alpha\|T\|\beta\rangle. \quad (5.16)$$

A more significant property of the states $\|\alpha\rangle$ is that they are given by the expansion

$$\|\alpha\rangle=\sum_n\frac{\alpha^n}{(n!)^{1/2}}|n\rangle \quad (5.17)$$

and thus obey the relation

$$a^\dagger\|\alpha\rangle=\frac{\partial}{\partial\alpha}\|\alpha\rangle. \quad (5.18)$$

While the properties of the alternatively normalized states $\|\alpha\rangle$ are worth bearing in mind, we have chosen not to adopt this normalization in the present paper in order to retain the more conventional interpretation of

scalar products as probability amplitudes. The advantage afforded by the relation (5.18) is not a great one since all of the operators we shall have to deal with are either already in normally ordered form, or easily so ordered.

VI. GENERAL PROPERTIES OF THE DENSITY OPERATOR

The formalism we have developed in the two preceding sections has been intended to provide a background for the expression of the density operator of a mode in terms of the vectors that represent coherent states. Viewed in mathematical terms, the use of the coherent state vectors in this way leads to considerable simplification in the calculation of statistical averages. The fact that these states are eigenstates of the field operators $\mathbf{E}^{(\pm)}(rt)$ means that normally ordered products of the field operators, when they are to be averaged, may be replaced by the products of their eigenvalues, i.e., treated not as operators, but as numbers. The field correlation functions such as $G^{(1)}$ given by Eq. (2.1) are averages of just such operator products. Their evaluation may be carried out quite conveniently through use of the representations we shall discuss.

Any density operator ρ may, according to the methods of the preceding section, be represented in a unique way by means of a function of two complex variables, $R(\alpha^*,\beta)$, which is analytic throughout the finite α^* and β planes. The function R is given explicitly, by means of Eq. (5.7), as

$$R(\alpha^*,\beta)=\langle\alpha|\rho|\beta\rangle\exp[\tfrac{1}{2}|\alpha|^2+\tfrac{1}{2}|\beta|^2]. \tag{6.1}$$

If we happen to know the matrix representation of ρ in the basis formed by the n-quantum states, the function R is evidently given by

$$R(\alpha^*,\beta)=\sum_{n,m}\langle n|\rho|m\rangle(n!m!)^{-1/2}(\alpha^*)^n\beta^m. \tag{6.2}$$

If we do not know the matrix elements $\langle n|\rho|m\rangle$ they may be found quite simply from a knowledge of $R(\alpha^*,\beta)$. One method for finding them is to consider $R(\alpha^*,\beta)$ as a generating function and identify its Taylor series with the series (6.2). A second method is to note that if we multiply Eq. (6.2) by $\alpha^i(\beta^*)^j\exp[-(|\alpha|^2+|\beta|^2)]$ and integrate over the α and β planes, then all terms save that for $n=i$ and $m=j$ vanish in the sum on the right and we have

$$\langle i|\rho|j\rangle=\frac{1}{\pi^2}\int R(\alpha^*,\beta)(i!j!)^{-1/2}\alpha^i(\beta^*)^j e^{-(|\alpha|^2+|\beta|^2)}d^2\alpha d^2\beta. \tag{6.3}$$

Given the knowledge of $R(\alpha^*,\beta)$, we may write the density operator as

$$\rho=\frac{1}{\pi^2}\int|\alpha\rangle R(\alpha^*,\beta)\langle\beta|e^{-\frac{1}{2}(|\alpha|^2+|\beta|^2)}d^2\alpha d^2\beta. \tag{6.4}$$

The statistical average of an operator T is given by the trace of the product ρT. If we calculate this average by using the representation (6.4) for ρ we must note that the trace of the expression $|\alpha\rangle\langle\beta|T$, regarded as an operator, is the matrix element $\langle\beta|T|\alpha\rangle$. Then, if we express the matrix element in terms of the function $\mathcal{T}(\alpha^*,\beta)$ defined by Eq. (5.7) we find

$$\mathrm{tr}\{\rho T\}=\frac{1}{\pi^2}\int R(\alpha^*,\beta)\mathcal{T}(\beta^*,\alpha)e^{-|\alpha|^2-|\beta|^2}d^2\alpha d^2\beta. \tag{6.5}$$

If T is any operator of the form $(a^\dagger)^n a^m$, its representation $\mathcal{T}(\beta^*,\alpha)$ is given by Eq. (5.11). In particular for $n=m=0$, we have the unit operator $T=1$ which is represented by $\mathcal{T}(\beta^*,\alpha)=\exp[\beta^*\alpha]$. Hence, the trace of ρ itself, which must be normalized to unity, is

$$\mathrm{tr}\rho=1$$
$$=\frac{1}{\pi^2}\int R(\alpha^*,\beta)\exp[\beta^*\alpha-|\alpha|^2-|\beta|^2]d^2\alpha d^2\beta.$$

Since $R(\alpha^*,\beta)$ is an entire function of α^*, we may use Eq. (4.10) to carry out the integration over the α plane. In this way we see that the normalization condition on R is

$$\frac{1}{\pi}\int R(\beta^*,\beta)e^{-|\beta|^2}d^2\beta=1. \tag{6.6}$$

The density operator is Hermitian and hence has real eigenvalues. These eigenvalues may be interpreted as probabilities and so must be positive numbers. Since ρ is thus a positive definite operator, its expectation value in any state, e.g., the state $|f\rangle$ defined by Eq. (4.6), must be non-negative,

$$\langle f|\rho|f\rangle\geq 0. \tag{6.7}$$

If, for example, we choose the state $|f\rangle$ to be a coherent state $|\alpha\rangle$ we find that the function R, which is given by Eq. (6.1), satisfies the inequality

$$R(\alpha^*,\alpha)\geq 0. \tag{6.8}$$

If we let the state $|f\rangle$ be specified as in Eq. (4.7) by an entire function $f(\alpha^*)$, then we find from the inequality (6.7) the more general condition for positive definiteness

$$\int[f(\alpha^*)]^* f(\beta^*)R(\alpha^*,\beta)e^{-|\alpha|^2-|\beta|^2}d^2\alpha d^2\beta\geq 0, \tag{6.9}$$

which must hold for all entire functions f.

In many types of physical experiments, particularly those dealing with fields which oscillate at extremely high frequencies, we cannot be said to have any *a priori* knowledge of the time-dependent parameters. The predictions we make in such circumstances are unchanged by displacements in time. They may be derived from a density operator which is stationary, that is, one

which commutes with the Hamiltonian operator or, more simply, with $a^\dagger a$. The necessary and sufficient condition that a function $R(\alpha^*,\beta)$ correspond to a stationary density operator is that it depend only on the product of its two variables, $\alpha^*\beta$. There must, in other words, exist an analytic function $\mathcal{S}$ such that

$$R(\alpha^*,\beta)=\mathcal{S}(\alpha^*\beta). \tag{6.10}$$

That this condition is a sufficient one is clear from the invariance of R under the multiplication of both α and β by a phase factor, $e^{i\varphi}$. The condition may be derived as a necessary one directly from the vanishing of the commutator of ρ with $a^\dagger a$. An alternative and perhaps simpler way of seeing the result depends on noting that a stationary ρ can only be a function of the Hamiltonian for the mode, or of $a^\dagger a$. It is therefore diagonal in the basis formed by the n-quantum states, i.e., $\langle n|\rho|m\rangle = \delta_{nm}\langle n|\rho|n\rangle$. Examination of the series expansion (6.2) for R then shows that it then takes the form of Eq. (6.10).

VII. THE P REPRESENTATION OF THE DENSITY OPERATOR

In the preceding sections we have demonstrated the generality of the use of the coherent states as a basis. Not all fields require for their description density operators of quite so general a form. Indeed for a broad class of radiation fields which includes, as we shall see, virtually all of those studied in optics, it becomes possible to reduce the density operator to a considerably simpler form. This form is one which brings to light many similarities between quantum electrodynamical calculations and the corresponding classical ones. Its use offers deep insights into the reasons why some of the fundamental laws of optics, such as those for superposition of fields and calculation of the resulting intensities, are the same as in classical theory, even when very few quanta are involved. We shall continue, for the present, to limit consideration to a single mode of the field.

One type of oscillator state which interests us particularly is, of course, a coherent state. The density operator for a pure state $|\alpha\rangle$ is just the projection operator

$$\rho=|\alpha\rangle\langle\alpha|. \tag{7.1}$$

The unique representation of this operator as a function $R(\beta^*,\gamma)$ is easily shown, from Eq. (6.1), to be

$$R(\beta^*,\gamma)=\exp[\beta^*\alpha+\gamma\alpha^*-|\alpha|^2]. \tag{7.2}$$

Other functions $R(\beta^*,\gamma)$, which satisfy the analyticity requirements necessary for the representations of density operators, may be constructed by forming linear combinations of exponentials such as (7.2) for various values of the complex parameter α. The functions R, which we form in this way, represent statistical mixtures of the coherent states. The most general such function R may be written as

$$R(\beta^*,\gamma)=\int P(\alpha)\exp[\beta^*\alpha+\gamma\alpha^*-|\alpha|^2]d^2\alpha, \tag{7.3}$$

where $P(\alpha)$ is a weight function defined at all points of the complex α plane. Since $R(\beta^*,\gamma)$ must satisfy the Hermiticity condition, Eq. (5.10), we require that the weight function be real-valued, i.e., $[P(\alpha)]^*=P(\alpha)$. The function $P(\alpha)$ need not be subject to any regularity conditions, but its singularities must be integrable ones.[11] It is convenient to allow $P(\alpha)$ to have delta-function singularities so that we may think of a pure coherent state as represented by a special case of Eq. (7.3). A real-valued two-dimensional delta function which is suited to this purpose may be defined as

$$\delta^{(2)}(\alpha)=\delta(\mathrm{Re}\,\alpha)\delta(\mathrm{Im}\,\alpha). \tag{7.4}$$

The pure coherent state $|\beta\rangle$ is then evidently described by

$$P(\alpha)=\delta^{(2)}(\alpha-\beta), \tag{7.5}$$

and the ground state of the oscillator is specified by setting $\beta=0$.

The density operator ρ which corresponds to Eq. (7.3) is just a superposition of the projection operators (7.1),

$$\rho=\int P(\alpha)|\alpha\rangle\langle\alpha|d^2\alpha. \tag{7.6}$$

It is the kind of operator we might naturally be led to if we were given knowledge that the oscillator is in a coherent state, but one which corresponds to an unknown eigenvalue α. The function $P(\alpha)$ might then be thought of as playing a role analogous to a probability density for the distribution of values of α over the complex plane.[12] Such an interpretation may, as we shall see, be justified at times. In general, however, it is not possible to interpret the function $P(\alpha)$ as a probability distribution in any precise way since the projection operators $|\alpha\rangle\langle\alpha|$ with which it is associated are not orthogonal to one another for different values of α. There is an approximate sense, as we have noted in connection with Eq. (3.33), in which two states $|\alpha\rangle$ and $|\alpha'\rangle$ may be said to become orthogonal to one another for $|\alpha-\alpha'|\gg 1$, i.e., when their wave packets (3.29) and those of the form (3.30) do not appreciably overlap. When the function $P(\alpha)$ tends to vary little over such large ranges of the parameter α, the nonorthogonality of the coherent states will make little difference, and $P(\alpha)$ will then be interpretable approximately as a probability density. The functions $P(\alpha)$

[11] If the singularities of $P(\alpha)$ are of types stronger than those of delta functions, e.g., derivatives of delta functions, the field represented will have no classical analog.

[12] The existence of this form for the density operator has also been observed by E. C. G. Sudarshan, Phys. Rev. Letters **10**, 277 (1963). His note is discussed briefly at the end of Sec. X.

which vary this slowly will, in general, be associated with strong fields, ones which may be described approximately in classical terms.

We shall call the expression (7.6) for the density operator the P representation in order to distinguish it from the more general form based on the functions R discussed earlier. The normalization property of the density operator requires that $P(\alpha)$ obey the normalization condition

$$\mathrm{tr}\rho=\int P(\alpha)d^2\alpha=1. \tag{7.7}$$

It is interesting to examine the conditions that the positive definiteness of ρ places upon $P(\alpha)$. If we apply the condition (6.9) to the function $R(\beta^*,\gamma)$ given by Eq. (7.3) we find

$$\int[f(\beta^*)]^*f(\gamma^*)P(\alpha)\exp[\beta^*\alpha+\gamma\alpha^*-|\alpha|^2-|\beta|^2-|\gamma|^2]\times d^2\alpha d^2\beta d^2\gamma\geq 0. \tag{7.8}$$

The γ integration may be carried out via Eq. (4.10) and the β integration by means of its complex conjugate. We then have the condition that

$$\int|f(\alpha^*)|^2P(\alpha)e^{-|\alpha|^2}d^2\alpha\geq 0 \tag{7.9}$$

must hold for all entire functions $f(\alpha^*)$. In particular, the choice $f(\alpha^*)=\exp[\beta\alpha^*-\frac{1}{2}|\beta|^2]$ leads to the simple condition

$$\int P(\alpha)e^{-|\alpha-\beta|^2}d^2\alpha\geq 0, \tag{7.10}$$

which must hold for all complex values of β. It corresponds to the requirement $\langle\beta|\rho|\beta\rangle\geq 0$. These conditions are immediately satisfied if $P(\alpha)$ is positive valued as it would be, were it a probability density. They are not strong enough, however, to exclude the possibility that $P(\alpha)$ takes on negative values over some suitably restricted regions of the plane.[13] This result serves to underscore the fact that the weight function $P(\alpha)$ cannot, in general, be interpreted as a probability density.[14]

If a density operator is specified by means of the P representation, its matrix elements connecting the n-quantum states are given by

$$\langle n|\rho|m\rangle=\int P(\alpha)\langle n|\alpha\rangle\langle\alpha|m\rangle d^2\alpha. \tag{7.11}$$

When Eqs. (3.3) and (3.6) are used to evaluate the scalar products in the integrand we find

$$\langle n|\rho|m\rangle=(n!m!)^{-1/2}\int P(\alpha)\alpha^n(\alpha^*)^me^{-|\alpha|^2}d^2\alpha. \tag{7.12}$$

This form for the density matrix indicates a fundamental property of the fields which are most naturally described by means of the P representation. If $P(\alpha)$ is a weight function with singularities no stronger than those of delta function type, it will, in general, possess nonvanishing complex moments of arbitrarily high order. [The unique exception is the choice $P(\alpha)=\delta^{(2)}(\alpha)$ which corresponds to the ground state of the mode.] It follows then that the diagonal matrix elements $\langle n|\rho|n\rangle$, which represent the probabilities for the presence of n photons in the mode, take on nonvanishing values for arbitrarily large n. There is thus no upper bound to the number of photons present when the function P is well behaved in the sense we have noted.[15]

Stationary density operators correspond in the P representation to functions $P(\alpha)$ which depend only on $|\alpha|$. This correspondence is made clear by Eq. (7.2) which shows that such $P(\alpha)$ lead to functions $R(\beta^*,\gamma)$ which are unaltered by a common phase change of β and γ. It is seen equally well through Eq. (7.12) which shows that $\langle n|\rho|m\rangle$ reduces to diagonal form when the weight function $P(\alpha)$ is circularly symmetric.

Some indication of the importance, in practical terms, of the P representation for the density operator can be found by considering the way in which photon fields produced by different sources become superposed. Since we are only discussing the behavior of one mode of the field for the present, we are only dealing with a fragment of the full problem, but all the modes may eventually be treated similarly. We shall illustrate the superposition law by assuming there are two different transient radiation sources coupled to the field mode and that they may be switched on and off separately. The first source will be assumed, when it is turned on alone at time t_1, to excite the mode from its ground state $|0\rangle$ to the coherent state $|\alpha_1\rangle$. If we assume that the source has ceased radiating by a time t_2, the state of the field remains $|\alpha_1\rangle$ for all later times. We may alternatively consider the case in which the first source remains inactive and the second one is switched on at

[13] An example of a weight function $P(\alpha)$ which takes on negative values but leads to a positive-definite density operator is given by the form

$$P(\alpha)=(1+\lambda)(\pi n)^{-1}\exp[-|\alpha|^2/n]-\lambda\delta^{(2)}(\alpha)$$

for $n>0$ and $0<\lambda<n^{-1}$. The matrix representation of the corresponding density operator, which is given by Eq. (7.12), is seen to be diagonal and to have only positive eigenvalues.

[14] A familiar example of a function which plays a role analogous to that of a probability density, but may take on negative values in quantum-mechanical contexts is the Wigner distribution function, E. P. Wigner, Phys. Rev. 40, 749 (1932).

[15] Density operators for fields in which the number of photons present possesses an upper bound N are represented by functions $R(\beta^*,\gamma)$ which are polynomials of Nth degree in β^* and in γ. It is evident from the behavior of such polynomials for large $|\beta|$ and $|\gamma|$ that any weight function $P(\alpha)$ which corresponds to $R(\beta^*,\gamma)$ through Eq. (7.2) would have to have singularities much stronger than those of a delta function. Such fields are probably represented more conveniently by means of the R function.

time t_2. The second source will then be assumed to bring the mode from its ground state to the coherent state $|\alpha_2\rangle$. We now ask what state the mode will be brought to if the two sources are allowed to act in succession, the first at t_1 and the second at t_2.

The answer for this simple case may be seen without performing any detailed calculations by making use of the unitary displacement operators described in Sec. III. The action of the first source is represented by the unitary operator $D(\alpha_1)$ which displaces the oscillator state from the ground state to the coherent state $|\alpha_1\rangle=D(\alpha_1)|0\rangle$. The action of the second source is evidently represented by the displacement operator $D(\alpha_2)$, so that when it is turned on after the first source, it brings the oscillator to the superposed state

$$|\ \rangle=D(\alpha_2)D(\alpha_1)|0\rangle. \tag{7.13}$$

Since the displacement operators are of the exponential form (3.17), their multiplication law is given by Eq. (3.20). We thus find

$$D(\alpha_2)D(\alpha_1)=D(\alpha_1+\alpha_2)\exp[\tfrac{1}{2}(\alpha_2\alpha_1^*-\alpha_2^*\alpha_1)]. \tag{7.14}$$

The exponential which has been separated from the D operators in this relation has a purely imaginary argument and, hence, corresponds to a phase factor. The superposed state, (7.13), in other words, is just the coherent state $|\alpha_1+\alpha_2\rangle$ multiplied by a phase factor. The phase factor has no influence upon the density operator for the superposed state, which is

$$\rho=|\alpha_1+\alpha_2\rangle\langle\alpha_1+\alpha_2|\,. \tag{7.15}$$

To vary the way in which the sources are turned on in the imaginary experiment we have described, e.g., to turn the two sources on at other times or in the reverse order, would only alter the final state through a phase factor and would thus lead to the same final density operator. The amplitudes of successive coherent excitations of the mode add as complex numbers in quantum theory, just as they do in classical theory.

Let us suppose next that the sources in the same experiment are somewhat less ideal and that, instead of exciting the mode to pure coherent states, they excite it to conditions described by mixtures of coherent states of the form (7.6). The first source acting alone, we assume, brings the field to a condition described by the density operator

$$\rho_1=\int P_1(\alpha_1)|\alpha_1\rangle\langle\alpha_1|\,d^2\alpha_1. \tag{7.16}$$

The condition produced by the second source, when it acts alone, is assumed to be represented by

$$\rho=\int P_2(\alpha_2)|\alpha_2\rangle\langle\alpha_2|\,d^2\alpha_2,$$

$$=\int P_2(\alpha_2)D(\alpha_2)|0\rangle\langle 0|D^{-1}(\alpha_2)d^2\alpha_2.$$

If the second source is turned on after the first, it brings the field to a condition described by the density operator

$$\rho=\int P_2(\alpha_2)D(\alpha_2)\rho_1 D^{-1}(\alpha_2)d^2\alpha_2,$$

$$=\int P_2(\alpha_2)P_1(\alpha_1)|\alpha_1+\alpha_2\rangle\langle\alpha_1+\alpha_2|\,d^2\alpha_1 d^2\alpha_2. \tag{7.17}$$

The latter density operator may be written in the general form

$$\rho=\int P(\alpha)|\alpha\rangle\langle\alpha|\,d^2\alpha,$$

if we define the weight function $P(\alpha)$ for the superposed excitations to be

$$P(\alpha)=\int\delta^{(2)}(\alpha-\alpha_1-\alpha_2)P_1(\alpha_1)P_2(\alpha_2)d^2\alpha_1 d^2\alpha_2, \tag{7.18}$$

$$=\int P_1(\alpha-\alpha')P_2(\alpha')d^2\alpha'. \tag{7.19}$$

We see immediately from Eq. (7.18) that P is correctly normalized if P_1 and P_2 are. The simple convolution law for combining the weight functions is one of the unique features of the description of fields by means of the P representation. It is quite analogous to the law we would use in classical theory to describe the probability distribution of the sum of two uncertain Fourier amplitudes for a mode.

The convolution theorem can often be used to separate fields into component fields with simpler properties. Suppose we have a field described by a weight function $P(\alpha)$ which has a mean value of α given by

$$\bar{\alpha}=\int\alpha P(\alpha)d^2\alpha. \tag{7.20}$$

It is clear from Eq. (7.19) that any such field may be regarded as the sum of a pure coherent field which corresponds to the weight function $\delta^{(2)}(\alpha-\bar{\alpha})$ and an additional field represented by $P(\alpha+\bar{\alpha})$ for which the mean value of α vanishes. Fields with vanishing mean values of α will be referred to as unphased fields.

The use of the P representation of the density operator, where it is not too singular, leads to simplifications in the calculation of statistical averages which go somewhat beyond those discussed in the last section. Thus, for example, the statistical average of any normally ordered product of the creation and annihilation operators, such as $(a^\dagger)^n a^m$, reduces to a simple average of $(\alpha^*)^n\alpha^m$ taken with respect to the weight

function $P(\alpha)$, i.e., we have

$$\mathrm{tr}\{\rho(a^\dagger)^n a^m\} = \int P(\alpha)\langle\alpha|(a^\dagger)^n a^m|\alpha\rangle d^2\alpha,$$

$$= \int P(\alpha)(\alpha^*)^n \alpha^m d^2\alpha. \tag{7.21}$$

This identity means, in practice, that many quantum-mechanical calculations can be carried out by means which are analogous to those already familiar from classical theory.

The mean number of photons which are present in a mode is the most elementary measure of the intensity of its excitation. The operator which represents the number of photons present is seen from Eq. (2.18) to be $a^\dagger a$. The average photon number, written as $\langle n\rangle$, is therefore given by

$$\langle n\rangle = \mathrm{tr}\{\rho a^\dagger a\}. \tag{7.22}$$

According to Eq. (7.21), with its two exponents set equal to unity, we have

$$\langle n\rangle = \int P(\alpha)|\alpha|^2 d^2\alpha, \tag{7.23}$$

i.e., the average photon number is just the mean squared absolute value of the amplitude α. When two fields described by distributions P_1 and P_2 are superposed, the resulting intensities are found from rules of the form which have always been used in classical electromagnetic theory. For unphased fields the intensities add "incoherently"; for coherent states the amplitudes add "coherently."

The use of the P representation of the density operator in describing fields brings many of the results of quantum electrodynamics into forms similar to those of classical theory. While these similarities make applications of the correspondence principle particularly clear, they must not be interpreted as indicating that classical theory is any sort of adequate substitute for the quantum theory. The weight functions $P(\alpha)$ which occur in quantum theoretical applications are not accurately interpretable as probability distributions, nor are they derivable as a rule from classical treatments of the radiation sources. They depend upon Planck's constant, in general, in ways that are unfathomable by classical or semiclassical analysis.

Since a number of calculations having to do with photon statistics have been carried out in the past by essentially classical methods, it may be helpful to discuss the relation between the P representation and the classical theory a bit further. It is worth noting in particular that the definition we have given the amplitude α as an eigenvalue of the annihilation operator is an intrinsically quantum-mechanical one. If we wish to represent a given classical field amplitude for the mode as an eigenvalue, then we see from Eq. (2.20) that the appropriate value of α has a magnitude which is proportional to $\hbar^{-1/2}$. In the dimensionless terms in which α is defined, the classical description of the mode only applies to the region $|\alpha|\gg 1$ of the complex α plane, i.e., to amplitudes of oscillation which are large compared with the range of the zero-point fluctuations present in the wave packet (3.29) and (3.30). Classical theory can therefore, in principle, only furnish us with the grossest sort of information about the weight function $P(\alpha)$. When the weight function extends appreciably into the classical regions of the plane, classical theory can only be relied upon, crudely speaking, to tell us average values of the function $P(\alpha)$ over areas whose dimensions, $|\Delta\alpha|$, are of order unity or larger. From Eq. (7.10) we see that such average values will always be positive; in the classical limit they may always be interpreted as probabilities.

VIII. THE GAUSSIAN DENSITY OPERATOR

The Gaussian function is a venerable statistical distribution, familiar from countless occurrences in classical statistics. We shall indicate in this section that it has its place in quantum field theory as well, where it furnishes the natural description of the most commonly occurring type of incoherence.[1]

Let us assume that the field mode we are studying is coupled to a number of sources which are essentially similar but are statistically independent of one another in their behavior. Such sources might, in practice, simply be several hypothetical subdivisions of one large source. If we may represent the contribution of each source (numbered $j=1, \cdots N$) to the excitation of the mode by means of a weight function $p(\alpha_j)$, we may then construct the weight function $P(\alpha)$ which describes the superposed fields by means of the generalized form of the convolution theorem

$$P(\alpha) = \int \delta^{(2)}\left(\alpha - \sum_{j=1}^{N}\alpha_j\right)\prod_{j=1}^{N} p(\alpha_j)d^2\alpha_j. \tag{8.1}$$

Since the weight functions which appear in this expression are all real valued, it is sometimes convenient to think of the amplitudes α in their arguments not as complex numbers, but as two-dimensional real vectors $\boldsymbol{\alpha}$ (i.e., $\boldsymbol{\alpha}_x = \mathrm{Re}\,\alpha$, $\boldsymbol{\alpha}_y = \mathrm{Im}\,\alpha$). Then if λ is an arbitrary complex number represented by the vector $\boldsymbol{\lambda}$, we may use a two-dimensional scalar product for the abbreviation

$$\mathrm{Re}\,\lambda\,\mathrm{Re}\,\alpha + \mathrm{Im}\,\lambda\,\mathrm{Im}\,\alpha = \boldsymbol{\alpha}\cdot\boldsymbol{\lambda}. \tag{8.2}$$

Using this notation, we may define the two-dimensional Fourier transform of the weight function $p(\boldsymbol{\alpha})$ as

$$\xi(\boldsymbol{\lambda}) = \int \exp(i\boldsymbol{\lambda}\cdot\boldsymbol{\alpha})p(\boldsymbol{\alpha})d^2\alpha. \tag{8.3}$$

The superposition law (8.1) then shows that the Fourier transform of the weight function $P(\boldsymbol{\alpha})$ is given by

$$\Xi(\boldsymbol{\lambda})=\int \exp(i\boldsymbol{\lambda}\cdot\boldsymbol{\alpha})P(\boldsymbol{\alpha})d^2\alpha,$$

$$=[\xi(\boldsymbol{\lambda})]^N. \tag{8.4}$$

If the individual sources are stationary ones their weight function $p(\boldsymbol{\alpha})$ depends only on $|\boldsymbol{\alpha}|$. The transform $\xi(\boldsymbol{\lambda})$ may then be approximated for small values of $|\boldsymbol{\lambda}|$ by

$$\xi(\boldsymbol{\lambda})=1-\tfrac{1}{4}\boldsymbol{\lambda}^2\int|\boldsymbol{\alpha}|^2p(\boldsymbol{\alpha})d^2\alpha,$$

$$=1-\tfrac{1}{4}\boldsymbol{\lambda}^2\langle|\boldsymbol{\alpha}|^2\rangle. \tag{8.5}$$

For values of $|\boldsymbol{\lambda}|$ which are smaller still (i.e., $|\boldsymbol{\lambda}|^2<N^{-1/2}\langle|\boldsymbol{\alpha}|^2\rangle^{-1}$), the transform Ξ for the superposed field may be approximated by

$$\Xi(\boldsymbol{\lambda})\approx\exp\{-\tfrac{1}{4}\boldsymbol{\lambda}^2N\langle|\boldsymbol{\alpha}|^2\rangle\}. \tag{8.6}$$

Since the weight function $p(\boldsymbol{\alpha})$ may take on negative values it is necessary at this point to verify that the second moment $\langle|\boldsymbol{\alpha}|^2\rangle$ is positive. That it is indeed positive is indicated by Eqs. (7.22) and (7.23) which show that $\langle|\boldsymbol{\alpha}|^2\rangle$ is the mean number of photons which would be radiated by each source in the absence of the others. For large values of N the transform $\Xi(\boldsymbol{\lambda})$ therefore decreases rapidly as $|\boldsymbol{\lambda}|$ increases. Since the function becomes vanishingly small for $|\boldsymbol{\lambda}|$ lying outside the range of approximation noted earlier, we may use (8.6) more generally as an asymptotic approximation to $\Xi(\boldsymbol{\lambda})$ for large N. When we calculate the transform of this asymptotic expression for $\Xi(\boldsymbol{\lambda})$ we find

$$P(\boldsymbol{\alpha})=(2\pi)^{-2}\int\exp(-i\boldsymbol{\alpha}\cdot\boldsymbol{\lambda})\Xi(\boldsymbol{\lambda})d^2\lambda,$$

$$=\frac{1}{\pi N\langle|\boldsymbol{\alpha}|^2\rangle}\exp(-\boldsymbol{\alpha}^2/N\langle|\boldsymbol{\alpha}|^2\rangle). \tag{8.7}$$

The mean value of $|\boldsymbol{\alpha}|^2$ for such a weight function is evidently $N\langle|\boldsymbol{\alpha}|^2\rangle$, but by the general theorem expressed in Eq. (7.23), this mean value is just the average of the total number of quanta present in the mode. If we write the latter average as $\langle n\rangle$, and resume the use of the complex notation for the variable α, the weight function (8.7) may be written as

$$P(\alpha)=\frac{1}{\pi\langle n\rangle}e^{-|\alpha|^2/\langle n\rangle}. \tag{8.8}$$

The weight function $P(\alpha)$ is positive everywhere and takes the same form as the probability distribution for the total displacement which results from a random walk in the complex plane. However, because the coherent states $|\alpha\rangle$ are not an orthogonal set, $P(\alpha)$ can only be accurately interpreted as a probability distribution for $\langle n\rangle\gg1$. We may note that it is not ultimately necessary, in order to derive Eq. (8.8), to assume that the weight functions corresponding to the individual sources are all the same. All that is required to carry out the proof is that the moments of the individual functions be of comparable magnitudes. The mean squared value of $|\alpha|$ is then given more generally by $\sum_j\langle|\alpha_j|^2\rangle$, rather than the value in Eq. (8.7), but this value is still the mean number of quanta in the mode, as indicated in Eq. (8.8).

It should be clear from the conditions of the derivation that the Gaussian distribution $P(\alpha)$ for the excitation of a mode possesses extremely wide applicability. The random or chaotic sort of excitation it describes is presumably characteristic of most of the familiar types of noncoherent macroscopic light sources, such as gas discharges, incandesant radiators, etc.

The Gaussian density operator

$$\rho=\frac{1}{\pi\langle n\rangle}\int e^{-|\alpha|^2/\langle n\rangle}|\alpha\rangle\langle\alpha|d^2\alpha \tag{8.9}$$

may be seen to take on a very simple form as well in the basis which specifies the photon numbers. To find this form we substitute in Eq. (8.9) the expansions (3.7) and (3.8) for the coherent states and note the identity

$$\pi^{-1}(l!m!)^{-1/2}\int\exp[-C|\alpha|^2]\alpha^l(\alpha^*)^md^2\alpha=\delta_{lm}C^{-(m+1)},$$

which holds for $C>0$. If we write $C=(1+\langle n\rangle)/\langle n\rangle$ we then find

$$\rho=\frac{1}{1+\langle n\rangle}\sum_m\left\{\frac{\langle n\rangle}{1+\langle n\rangle}\right\}^m|m\rangle\langle m|. \tag{8.10}$$

In other words, the number of quanta in the mode is distributed according to the powers of the parameter $\langle n\rangle/(1+\langle n\rangle)$. The Planck distribution for blackbody radiation furnishes an illustration of a density operator which has long been known to take the form of Eq. (8.10). The thermal excitation which leads to the blackbody distribution is an ideal example of the random type we have described earlier, and so it should not be surprising that this distribution is one of the class we have derived. It is worth noting, in particular, that while the Planck distribution is characteristic of thermal equilibrium, no such limitation is implicit in the general form of the density operator (8.9). It will apply whenever the excitation has an appropriately random quality, no matter how far the radiator is from thermal equilibrium.

The Gaussian distribution function $\exp[-|\alpha|^2/\langle n\rangle]$ is phrased in terms which are explicitly quantum mechanical. In the limit which would represent a classical field both $|\alpha|^2$ and the average quantum number $\langle n\rangle$ become infinite as $\hbar^{-1}$, but their quotient, which is the argument of the Gaussian function, remains

well defined. The form which the distribution takes in the classical limit is a familiar one. Historically, one of the origins of the random walk problem is to be found in the discussion of a classical harmonic oscillator which is subject to random excitations.[16] Such oscillators have complex amplitudes which are described under quite general conditions by a Gaussian distribution. If we were armed with this knowledge, and lacked the quantum-mechanical analysis given earlier, we might be tempted to assume that a Gaussian distribution derived in this way from classical theory can describe the photon distribution. To demonstrate the fallacy of this view we must examine more closely the nature of the parameter $\langle n\rangle$ which is, after all, the only physical constant involved in the distribution. We may take, as a simple illustration, the case of thermal excitation corresponding to temperature T. Then the mean photon number is given by $\langle n\rangle=[\exp(\hbar\omega/\kappa T)-1]^{-1}$, where κ is Boltzmann's constant, and the distribution $P(\alpha)$ takes the form

$$P(\alpha)=\frac{1}{\pi}[e^{\hbar\omega/\kappa T}-1]\exp[-(e^{\hbar\omega/\kappa T}-1)|\alpha|^2]. \quad (8.11)$$

To reach the classical analog of this distribution we would assume that the classical field energy in the mode, $H=\frac{1}{2}\int(\mathbf{E}^2+\mathbf{B}^2)d\mathbf{r}$, is distributed with a probability proportional to $\exp[-H/\kappa T]$. The distribution for the amplitude α that results is

$$P_{cl}(\alpha)=(\hbar\omega/\pi\kappa T)\exp[-\hbar\omega|\alpha|^2/\kappa T], \quad (8.12)$$

which is seen to be a first approximation in powers of $\hbar$ to the correct distribution. (Again, we must remember that the quantity $\hbar|\alpha|^2$ is to be construed as a classical parameter.) The distribution $P_{cl}(\alpha)$ only extends into the classical region of the plane, $|\alpha|\gg1$, for low-frequency modes, that is, only for $(\hbar\omega/\kappa T)\ll1$ are the modes sufficiently excited to be accurately described by classical theory. For higher frequencies the two distributions differ greatly in nature even though both are Gaussian. The classical distribution retains much too large a radius in the α plane as $\hbar\omega$ increases beyond κT, rather than narrowing extremely rapidly as the correct distribution does.[17] That error, in fact, epitomizes the ultraviolet catastrophe of the classical radiation theory. The example we have discussed is, of course, an elementary one, but it should serve to illustrate some of the points noted in the preceding section regarding the limitations of the classical distribution function.

The expression for the thermal density operator of an oscillator in terms of coherent quantum states appears to offer new and instructive approaches to many familiar problems. It permits us, for example, to derive the thermal averages of exponential functions of the operators a and $a^\dagger$ in an elementary way. The thermal average of the operator $D(\beta)$ defined by Eq. (3.17) is an illustration. It is given by

$$\mathrm{tr}\{\rho D(\beta)\}=\frac{1}{\pi\langle n\rangle}\int e^{-|\alpha|^2/\langle n\rangle}\langle\alpha|D(\beta)|\alpha\rangle d^2\alpha. \quad (8.13)$$

The expectation value in the integrand is, in this case

$$\begin{aligned}\langle\alpha|D(\beta)|\alpha\rangle&=\langle0|D^{-1}(\alpha)D(\beta)D(\alpha)|0\rangle,\\&=\exp[\beta\alpha^*-\beta^*\alpha]\langle0|D(\beta)|0\rangle,\\&=\exp[\beta\alpha^*-\beta^*\alpha]\langle0|\beta\rangle,\\&=\exp[\beta\alpha^*-\beta^*\alpha-\tfrac{1}{2}|\beta|^2], \quad (8.14)\end{aligned}$$

where the properties of $D(\alpha)$ as a displacement operator have been used in the intermediate steps. When the integration indicated in Eq. (8.13) is carried out, we find

$$\mathrm{tr}\{\rho D(\beta)\}=\exp[-|\beta|^2(\langle n\rangle+\tfrac{1}{2})], \quad (8.15)$$

which is a frequently used corollary of Bloch's theorem on the distribution function of an oscillator coordinate.[18]

IX. DENSITY OPERATORS FOR THE FIELD

The developments introduced in Secs. III–VIII have all concerned the description of the quantum state of a single mode of the electromagnetic field. We may describe the field as a whole by constructing analogous methods to deal with all its modes at once. For this purpose we introduce a basic set of coherent states for the entire field and write them as

$$|\{\alpha_k\}\rangle\equiv\prod_k|\alpha_k\rangle_k, \quad (9.1)$$

where the notation $\{\alpha_k\}$, which will be used in several other connections, stands for the set of all amplitudes α_k. It is clear then, from the arguments of Sec. IV, that any state of the field determines uniquely a function $f(\{\alpha_k^*\})$ which is an entire function of each of the variables α_k^*. If the Hilbert space vector which represents the state is known and designated as $|f\rangle$, the function f is given by

$$f(\{\alpha_k^*\})=\langle\{\alpha_k\}|f\rangle\exp(\tfrac{1}{2}\textstyle\sum_k|\alpha_k|^2), \quad (9.2)$$

which is the direct generalization of Eq. (4.11). The expansion for the state $|f\rangle$ in terms of coherent states is then

$$|f\rangle=\int|\{\alpha_k\}\rangle f(\{\alpha_k^*\})\prod_k\pi^{-1}e^{-\frac{1}{2}|\alpha_k|^2}d^2\alpha_k, \quad (9.3)$$

which generalizes Eq. (4.7).

All of the operators which occur in field theory possess expansions in terms of the vectors $|\{\alpha_k\}\rangle$ and their

[16] Lord Rayleigh, *The Theory of Sound*, (MacMillan and Company Ltd., London, 1894), 2nd ed., Vol. I, p. 35; *Scientific Papers* (Cambridge University Press, Cambridge, England, 1899–1920), Vol. I, p. 491, Vol. IV, p. 370.

[17] For frequencies in the middle of the visible spectrum and temperatures under 3000°K the quantum mechanical distribution (8.11) will have a radius which corresponds to $|\alpha|^2\ll10^{-3}$, i.e., the distribution is far from classical in nature. Comparable radii characterize the distributions for nonthermal incoherent sources.

[18] F. Bloch, Z. Physik 74, 295 (1932).

adjoints. To construct such representations is simply a matter of generalizing the formulas of Sec. V to deal with an infinite set of amplitude variables. We therefore proceed directly to a discussion of the density operator. For any density operator ρ we may define a function $R(\{\alpha_k^*\},\{\beta_k\})$ which is an entire function of each of the variables α_k^* and β_k for all modes k. This function, as may be seen from Eq. (6.1), is given by

$$R(\{\alpha_k^*\},\{\beta_k\})=\langle\{\alpha_k\}|\rho|\{\beta_k\}\rangle \times\exp[\tfrac{1}{2}\textstyle\sum(|\alpha_k|^2+|\beta_k|^2)]. \quad (9.4)$$

The corresponding representation of the density operator is

$$\rho=\int|\{\alpha_k\}\rangle R(\{\alpha_k^*\},\{\beta_k\})\langle\{\beta_k\}|\prod_k\pi^{-2} \times e^{-\frac{1}{2}(|\alpha_k|^2+|\beta_k|^2)}d^2\alpha_k d^2\beta_k. \quad (9.5)$$

If the set of integers $\{n_k\}$ is used to specify the familiar stationary states which have n_k photons in the kth mode, we may regard R as a generating function for the matrix elements of ρ connecting these states, i.e., as a generalization of Eq. (6.2) we have

$$R(\{\alpha_k^*\},\{\beta_k\})=\sum_{\{n_k\},\{m_k\}}\langle\{n_k\}|\rho|\{m_k\}\rangle \times\prod_k(n_k!m_k!)^{-1/2}(\alpha_k^*)^{n_k}\beta_k^{m_k}. \quad (9.6)$$

The matrix elements of ρ in the stationary basis are then given by

$$\langle\{n_k\}|\rho|\{m_k\}\rangle =\int R(\{\alpha_k^*\},\{\beta_k\})\prod_k\pi^{-2}(n_k!m_k!)^{-1/2}\alpha_k^{n_k}(\beta_k^*)^{m_k} \times e^{-(|\alpha_k|^2+|\beta_k|^2)}d^2\alpha_k d^2\beta_k. \quad (9.7)$$

The normalization condition on R is clearly

$$\int R(\{\beta_k^*\},\{\beta_k\})\prod_k\pi^{-1}e^{-|\beta_k|^2}d^2\beta_k=1. \quad (9.8)$$

The positive definiteness condition, Eq. (6.9), may also be generalized in an evident way to deal with the full set of amplitude variables.

It may help as a simple illustration of the foregoing formulae to consider the representation of a single-photon wave packet. The state which is empty of all photons is the one for which the amplitudes α_k all vanish. If we write that state as $|\text{vac}\rangle$, then we may write the most general one-photon state as $\sum_k q(k)a_k^\dagger|\text{vac}\rangle$, where the function $q(k)$ plays the role of a packet amplitude. The function f which represents this state is then

$$f(\{\alpha_k^*\})=\sum_k q(k)\alpha_k^*, \quad (9.9)$$

and the corresponding function R which determines the density operator is

$$R(\{\alpha_k^*\},\{\beta_k\})=\sum_k q(k)\alpha_k^*\sum_{k'}q^*(k')\beta_{k'}. \quad (9.10)$$

The normalization condition (9.8) corresponds to the requirement $\sum|q(k)|^2=1$. Since the state we have considered is a pure one, the function R factorizes into the product of two functions, one having the form of f and the other of its complex conjugate. If the packet amplitudes $q(k)$ were in some degree unpredictable, as they usually are, the packet could no longer be represented by a pure state. The function R would then be an average taken over the distribution of the amplitudes $q(k)$ and hence would lose its factorizable form in general. Whenever an upper bound exists for the number of photons present, i.e., the number of photons is required to be less than or equal to some integer N, we will find that R is a polynomial of at most Nth degree in the variables $\{\alpha_k^*\}$ and of the same degree in the $\{\beta_k\}$.

There will, of course, exist many types of excitation for which the photon numbers are unbounded. Among these are the ones which are more conveniently described by means of a generalized P distribution, i.e., the excitations for which there exists a reasonably well-behaved real-valued function $P(\{\alpha_k\})$ such that

$$R(\{\beta_k^*\},\{\gamma_k\})=\int P(\{\alpha_k\}) \times\exp\Big[\sum_k(\beta_k^*\alpha_k+\gamma_k\alpha_k^*-|\alpha_k|^2)\Big]\prod_k d^2\alpha_k. \quad (9.11)$$

When R possesses a representation of this type the density operator (9.5) may be reduced by means of Eq. (4.14) and its complex conjugate to the simple form

$$\rho=\int P(\{\alpha_k\})|\{\alpha_k\}\rangle\langle\{\alpha_k\}|\prod_k d^2\alpha_k, \quad (9.12)$$

which is the many-mode form of the P representation given by Eq. (7.6). The function P must satisfy the positive definiteness condition

$$\int|f(\{\alpha_k^*\})|^2P(\{\alpha_k\})\prod_k e^{-|\alpha_k|^2}d^2\alpha_k\geq 0 \quad (9.13)$$

for all possible choices of entire functions $f(\{\alpha_k^*\})$. The matrix elements of the density operator in the representation based on the n-photon states are

$$\langle\{n_k\}|\rho|\{m_k\}\rangle=\int P(\{\alpha_k\}) \times\prod_k(n_k!m_k!)^{-1/2}\alpha_k^{n_k}(\alpha_k^*)^{m_k}e^{-|\alpha_k|^2}d^2\alpha_k. \quad (9.14)$$

Stationary density operators, i.e., ones which commute with the Hamiltonian correspond to functions $P(\{\alpha_k\})$ which depend on the amplitude variables only through their magnitudes $\{|\alpha_k|\}$.

The superposition of two fields is described by forming the convolution integral of their distribution functions, much as in the case of a single mode. Thus, if two fields, described by $P_1(\{\beta_k\})$ and $P_2(\{\gamma_k\})$, respectively, are superposed, the resulting field has a distribution function

$$P(\{\alpha_k\})=\int \prod_k \delta^{(2)}(\alpha_k-\beta_k-\gamma_k) \times P_1(\{\beta_k\})P_2(\{\gamma_k\}) \prod_k d^2\beta_k d^2\gamma_k. \quad (9.15)$$

For fields which are represented by means of the density operator (9.12) all of the averages of normally ordered operator products can be calculated by means of formulas which, as in the case of a single mode, greatly resemble those of classical theory. Thus, the parameters $\{\alpha_k\}$ play much the same role in these calculations as the random Fourier amplitudes of the field do in the familiar classical theory of microwave noise.[19] Furthermore, the weight function $P(\{\alpha_k\})$ plays a role similar to that of the probability distribution for the Fourier amplitudes. Although this resemblance is extremely convenient in calculations, and offers immediate insight into the application of the correspondence principle, we must not lose sight of the fact that the function $P(\{\alpha_k\})$ is, in general, an explicitly quantum-mechanical structure. It may assume negative values, and is not accurately interpretable as a probability distribution except in the classical limit of strongly excited or low frequency fields.

In the foregoing discussions we have freely assumed that the density operator which describes the field is known and that it may, therefore, be expressed either in the representation of Eq. (9.5) or in the P representation of Eq. (9.12). For certain types of incoherent sources which we have discussed in Sec. VIII and will mention again in Sec. X, the explicit construction of these density operators is not at all difficult. But to find accurate density operators for other types of sources, including the recently developed coherent ones, will require a good deal of physical insight. The general problem of treating quantum mechanically the interaction of a many-atom source both with the radiation field and with an excitation mechanism of some sort promises to be a complicated one. It will have to be approached, no doubt, through greatly simplified models.

Since very little is known about the density operator for radiation fields, some insight may be gained by examining the form it takes on in one of the few completely soluble problems of quantum electrodynamics. We shall study the photon field radiated by an electric current distribution which is essentially classical in nature, one that does not suffer any noticeable reaction from the process of radiation. We may then represent the radiating current by a prescribed vector function of space and time $\mathbf{j}(\mathbf{r},t)$. The Hamiltonian which describes the coupling of the quantized electromagnetic field to the current distribution takes the form

$$H_1(t)=-\frac{1}{c}\int \mathbf{j}(\mathbf{r},t)\cdot\mathbf{A}(\mathbf{r},t)d\mathbf{r}. \quad (9.16)$$

The introduction of an explicitly time-dependent interaction of this type means that the state vector for the field, $|\ \rangle$, which previously was fixed (corresponding to the Heisenberg picture) will begin to change with time in accordance with the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}|\ \rangle=H_1(t)|\ \rangle, \quad (9.17)$$

which is the one appropriate to the interaction representation. The solution of this equation is easily found.[20] If we assume that the initial state of the field at time $t=-\infty$ is one empty of all photons, then the state of the field at time t may be written in the form

$$|t\rangle=\exp\left\{\frac{i}{\hbar c}\int_{-\infty}^{t}dt'\int \mathbf{j}(\mathbf{r},t')\cdot\mathbf{A}(\mathbf{r},t')d\mathbf{r}+i\varphi(t)\right\}|\mathrm{vac}\rangle. \quad (9.18)$$

The function $\varphi(t)$ which occurs in the exponent is a real-valued c-number phase function. It is easily evaluated, but cancels out of the product $|t\rangle\langle t|$ and so has no bearing on the construction of the density operator. The exponential operator which occurs in Eq. (9.18) may be expressed quite simply in terms of the displacement operators we discussed in Sec. III. For this purpose we define a displacement operator D_k for the kth mode as

$$D_k(\beta_k)=\exp[\beta_k\alpha_k^{\dagger}-\beta_k^{*}\alpha_k]. \quad (9.19)$$

Then it is clear from the expansion (2.10) for the vector potential that we may write

$$\exp\left\{\frac{i}{\hbar c}\int_{-\infty}^{t}dt'\int \mathbf{j}(\mathbf{r},t')\cdot\mathbf{A}(\mathbf{r},t')d\mathbf{r}\right\}=\prod_k D_k[\alpha_k(t)], \quad (9.20)$$

where the time-dependent amplitudes $\alpha_k(t)$ are given by

$$\alpha_k(t)=\frac{i}{(2\hbar\omega)^{1/2}}\int_{-\infty}^{t}dt'\int d\mathbf{r}\,\mathbf{u}_k^{*}(\mathbf{r})\cdot\mathbf{j}(\mathbf{r},t')e^{i\omega t'}. \quad (9.21)$$

The density operator at time t may therefore be written

[19] J. Lawson and G. E. Uhlenbeck, *Threshold Noise Signals* (McGraw-Hill Book Company, Inc., New York, 1950), pp. 33–56.

[20] R. J. Glauber, Phys. Rev. **84**, 395 (1951).

as

$$|t\rangle\langle t| = \prod_k D_k[\alpha_k(t)]|\text{vac}\rangle\langle\text{vac}|\prod_k D_k^{-1}[\alpha_k(t)] \quad (9.22)$$

$$= |\{\alpha_k(t)\}\rangle\langle\{\alpha_k(t)\}|\,. \quad (9.23)$$

The radiation by any prescribed current distribution, in other words, always leads to a pure coherent state.

It is only a slight generalization of the model we have just considered to imagine that the current distribution $\mathbf{j}(\mathbf{r},t)$ is not wholly predictable. In that case the amplitudes $\alpha_k(t)$ defined by Eq. (9.21) become random variables which possess collectively a probability distribution function which we may write as $p(\{\alpha_k\},t)$. The density operator for the field radiated by such a random current then becomes

$$\rho(t) = \int p(\{\alpha_k\},t)\,|\{\alpha_k\}\rangle\langle\{\alpha_k\}|\prod_k d^2\alpha_k. \quad (9.24)$$

We see that the density operator for a field radiated by a random current which suffers no recoil in the radiation process always takes the form of the P representation of Eq. (9.12). The weight function in this case does admit interpretation as a probability distribution, but it has a classical structure associated directly with the properties of the radiating current rather than with particular (nonorthogonal) states of the field. The assumption we have made in defining the model, that the current suffers negligible reaction, is a strong one but is fairly well fulfilled in radiating systems operated at radio or microwave frequencies. The fields produced by such systems should be accurately described by density operators of the form (9.24).

X. CORRELATION AND COHERENCE PROPERTIES OF THE FIELD

Any eigenvalue function $\mathcal{E}(\mathbf{r}t)$ which satisfies the appropriate field equations and contains only positive frequency terms determines a set of mode amplitudes $\{\alpha_k\}$ uniquely through the expansion (2.20). This set of mode amplitudes then determines a coherent state of the field, $|\{\alpha_k\}\rangle$, such that

$$\mathbf{E}^{(+)}(\mathbf{r}t)\,|\{\alpha_k\}\rangle = \mathcal{E}(\mathbf{r}t)\,|\{\alpha_k\}\rangle. \quad (10.1)$$

To discuss the general form which the field correlation functions take in such states it is convenient to abbreviate a set of coordinates $(\mathbf{r}_j,t_j)$ by a single symbol x_j. The nth-order correlation function is then defined as[3]

$$G_{\mu_1\cdots\mu_{2n}}{}^{(n)}(x_1\cdots x_{2n}) = \text{tr}\{\rho E_{\mu_1}{}^{(-)}(x_1)\cdots \times E_{\mu_n}{}^{(-)}(x_n)E_{\mu_{n+1}}{}^{(+)}(x_{n+1})\cdots E_{\mu_{2n}}{}^{(+)}(x_{2n})\}. \quad (10.2)$$

The density operator for the coherent state defined by Eq. (10.1) is the projection operator

$$\rho = |\{\alpha_k\}\rangle\langle\{\alpha_k\}|\,. \quad (10.3)$$

For this operator it follows from Eq. (10.1) and its Hermitian adjoint that the correlation functions reduce to the factorized form

$$G_{\mu_1\cdots\mu_{2n}}{}^{(n)}(x_1\cdots x_{2n}) = \prod_{j=1}^{n}\mathcal{E}_{\mu_j}{}^*(x_j)\prod_{l=n+1}^{2n}\mathcal{E}_{\mu_l}(x_l). \quad (10.4)$$

In other words, the field which corresponds to the state $|\{\alpha_k\}\rangle$ satisfies the conditions for full coherence according to the definition[3] given earlier.

It is worth noting that the state $|\{\alpha_k\}\rangle$ is not the only one which leads to the set of correlation functions (10.4). Indeed, let us consider a state which corresponds not to the amplitudes $\{\alpha_k\}$, but to a set $\{e^{i\varphi}\alpha_k\}$ which differs by a common phase factor (i.e., φ is real and independent of k). Then the corresponding eigenvalue function becomes $e^{i\varphi}\mathcal{E}(\mathbf{r}t)$, but such a change leaves the correlation functions (10.4) unaltered. It is clear from this invariance property of the correlation functions that certain mixtures of the coherent states also lead to the same set of functions. Thus, if $|\{\alpha_k\}\rangle$ is the state defined by Eq. (10.1), and $\mathfrak{L}(\varphi)$ is any real-valued function of φ normalized in the sense

$$\int_0^{2\pi}\mathfrak{L}(\varphi)d\varphi = 1, \quad (10.5)$$

we see that the density operator

$$\rho = \int_0^{2\pi}\mathfrak{L}(\varphi)\,|\{e^{i\varphi}\alpha_k\}\rangle\langle\{e^{i\varphi}\alpha_k\}|\,d\varphi \quad (10.6)$$

leads for all choices of $\mathfrak{L}(\varphi)$ to the set of correlation functions (10.4). Such a density operator is, of course, a special case of the general form (9.12), one which corresponds to an over-all uncertainty in the phase of the $\{\alpha_k\}$. The particular choice $\mathfrak{L}(\varphi) = (2\pi)^{-1}$, which corresponds to complete ignorance of the phase, represents the usual state of our knowledge about high-frequency fields. We have been careful, therefore, to define coherence in terms of a set of correlation functions which are independent of the over-all phase.

Since nonstationary fields of many sorts can be represented by means of eigenvalue functions, it becomes a simple matter to construct corresponding quantum states. As an illustration we may consider the example of an amplitude-modulated plane wave. For this purpose we make use of the particular set of mode functions defined by Eq. (2.9). Then if the carrier wave has frequency ω and the modulation is periodic and has frequency $\zeta\omega$ where $0<\zeta<1$, we may write an appropriate eigenvalue function as

$$\mathcal{E}(\mathbf{r}t) = i\left(\frac{\hbar\omega}{2L^3}\right)^{1/2}\hat{e}^{(\lambda)}\alpha_{\mathbf{k}} \times\{1+M\cos[\zeta(\mathbf{k}\cdot\mathbf{r}-\omega t)-\delta]\}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}. \quad (10.7)$$

When this expression is expanded in plane-wave modes it has only three nonvanishing amplitude coefficients. These are $\alpha_{\mathbf{k}}$ itself and the two sideband amplitudes

$$\alpha_{\mathbf{k}(1-\zeta)}=\tfrac{1}{2}M(1-\zeta)^{-1/2}e^{i\delta}\alpha_{\mathbf{k}},$$
$$\alpha_{\mathbf{k}(1+\zeta)}=\tfrac{1}{2}M(1+\zeta)^{-1/2}e^{-i\delta}\alpha_{\mathbf{k}}. \tag{10.8}$$

The coherent state which corresponds to the modulated wave may be constructed immediately from the knowledge of these amplitudes. In practice, of course, we will not often know the phase of $\alpha_{\mathbf{k}}$, and so the wave should be represented not by a single coherent state, but by a mixture of the form (10.6). Representations of other forms of modulated waves may be constructed similarly.

Incoherent fields, or the broad class of fields for which the correlation functions do not factorize, must be described by means of density operators which are more general in their structure than those of Eqs. (10.3) or (10.6). To illustrate the form taken by the correlation functions for such cases we may suppose the field to be described by the P representation of the density operator. Then the first-order correlation function is given by

$$G_{\mu\nu}^{(1)}(\mathbf{r}t,\mathbf{r}'t')=\int P(\{\alpha_k\})\sum_{k,k'}\tfrac{1}{2}\hbar(\omega\omega')^{1/2}u_{k\mu}^{*}(\mathbf{r})u_{k'\nu}(\mathbf{r}')$$
$$\times\alpha_k^{*}\alpha_{k'}e^{i(\omega t-\omega' t')}\prod_l d^2\alpha_l. \tag{10.9}$$

Fields for which the P representation is inconveniently singular may, as we have noted earlier, always be described by means of analytic functions $R(\{\alpha_k^{*}\},\{\beta_k\})$ and corresponding density operators of the form (9.5). When that form of density operator is used to evaluate the first-order correlation function we find

$$G_{\mu\nu}^{(1)}(\mathbf{r}t,\mathbf{r}'t')=\int R(\{\alpha_k^{*}\},\{\beta_k\})\sum_{k',k''}\tfrac{1}{2}\hbar(\omega'\omega'')^{1/2}$$
$$\times u_{k'\mu}^{*}(\mathbf{r})u_{k''\nu}(\mathbf{r}')\beta_{k'}^{*}\alpha_{k''}e^{i(\omega' t-\omega'' t')}$$
$$\times\prod_l e^{\beta_l^{*}\alpha_l}d\mu(\alpha_l)d\mu(\beta_l), \tag{10.10}$$

where the differentials $d\mu(\alpha_l)$ and $d\mu(\beta_l)$ are those defined by Eq. (5.14). The higher order correlation functions are given by integrals analogous to (10.9) and (10.10). Their integrands contain polynomials of the $2n$th degree in the amplitude variables α_k and β_k^{*} in place of the quadratic forms which occur in the first-order functions.

The energy spectrum of a radiation field is easily derived from a knowledge of its first-order correlation function. If we return for a moment to the expansion (2.19) for the positive-frequency field operator, and write the negative-frequency field as its Hermitian adjoint, we see that these operators obey the identity

$$2\int\mathbf{E}^{(-)}(\mathbf{r}t)\cdot\mathbf{E}^{(+)}(\mathbf{r}t')d\mathbf{r}$$
$$=\sum_k\hbar\omega a_k^{\dagger}a_k\exp[i\omega(t-t')]. \tag{10.11}$$

If we take the statistical average of both sides of this equation we may write the resulting relation as

$$\sum_\mu\int G_{\mu\mu}^{(1)}(\mathbf{r}t,\mathbf{r}t')d\mathbf{r}=\tfrac{1}{2}\sum_k\hbar\omega\langle n_k\rangle\exp[i\omega(t-t')], \tag{10.12}$$

where $\langle n_k\rangle$ is the average number of photons in the kth mode. The Fourier representation of the volume integral of $\sum_\mu G_{\mu\mu}^{(1)}$ therefore identifies the energy spectrum $\hbar\omega\langle n_k\rangle$ quite generally.

For fields which may be represented by stationary density operators, it becomes still simpler to extract the energy spectrum from the correlation function. For such fields the weight function $P(\{\alpha_k\})$ depends only on the absolute values of the α_k, so that we have

$$\int P(\{\alpha_k\})\alpha_{k'}^{*}\alpha_{k''}\prod_l d^2\alpha_l=\langle|\alpha_{k'}|^2\rangle\delta_{k'k''}$$
$$=\langle n_{k'}\rangle\delta_{k'k''}. \tag{10.13}$$

By using Eq. (10.9) to evaluate the correlation function, and specializing to the case of plane-wave modes, we then find

$$\sum_\mu G_{\mu\mu}^{(1)}(\mathbf{r}t,\mathbf{r}t')=\tfrac{1}{2}L^{-3}\sum_{\mathbf{k},\lambda}\hbar\omega\langle n_{\mathbf{k},\lambda}\rangle e^{i\omega(t-t')}, \tag{10.14}$$

in which we have explicitly indicated the role of the polarization index λ. If the volume which contains the field is sufficiently large in comparison to the wavelengths of the excited modes, the sum over the modes in Eq. (10.14) may be expressed as an integral over $\mathbf{k}$ space $[\sum_k\rightarrow\int L^3(2\pi)^{-3}d\mathbf{k}]$. By defining an energy spectrum for the quanta present (i.e., an energy per unit interval of ω) as

$$w(\omega)=(2\pi)^{-3}\hbar k^3\sum_\lambda\int\langle n_{\mathbf{k},\lambda}\rangle d\Omega_{\mathbf{k}}, \tag{10.15}$$

where $d\Omega_{\mathbf{k}}$ is an element of solid angle in $\mathbf{k}$ space, we may then rewrite Eq. (10.14) in the form

$$\sum_\mu G_{\mu\mu}^{(1)}(\mathbf{r}t,\mathbf{r}t')=\tfrac{1}{2}\int_0^\infty w(\omega)e^{i\omega(t-t')}d\omega. \tag{10.16}$$

With the understanding that $w(\omega)=0$ for $\omega<0$, we may extend the integral over ω from $-\infty$ to ∞. It is then clear that the relation (10.16) may be inverted to express the energy spectrum as the Fourier transform of

the time-dependent correlation function,

$$w(\omega)=\frac{1}{\pi}\int_{-\infty}^{\infty}\sum_{\mu}G_{\mu\mu}^{(1)}(\mathbf{r}0,\mathbf{r}t)e^{i\omega t}dt. \qquad (10.17)$$

A pair of relations analogous to Eqs. (10.16) and (10.17), and together called the Wiener-Khintchine theorem, has long been of use in the classical theory of random fields.[21] The relations we have derived are, in a sense, the natural quantum mechanical generalization of the Wiener-Khintchine theorem. All we have assumed is that the field is describable by a stationary form of the P representation of the density operator. The proof need not, in fact, rest upon the use of the P representation since we can construct a corresponding statement in terms of the more general representation (9.5).

Stationary fields, according to Eq. (6.10), are represented by entire functions $R=\mathcal{S}(\{\alpha_k^*\beta_k\})$, i.e., functions which depend only on the set of products $\alpha_k^*\beta_k$. For such fields, then, the integral over the α and β planes which is required in Eq. (10.10) takes the form

$$\langle\beta_{k'}^*\alpha_{k''}\rangle=\int\mathcal{S}(\{\alpha_k^*\beta_k\})\beta_{k'}^*\alpha_{k''}\prod_l e^{\beta_l^*\alpha_l}d\mu(\alpha_l)d\mu(\beta_l). \qquad (10.18)$$

Since the range of integration of each of the α and β variables covers the entire complex plane, this integral cannot be altered if we change the signs of any of the variables. If, however, we replace the particular variables $\alpha_{k''}$ and $\beta_{k''}$ by $-\alpha_{k''}$ and $-\beta_{k''}$ the integral is seen to reverse in sign, unless we have

$$\langle\beta_{k'}^*\alpha_{k''}\rangle=\delta_{k'k''}\langle\beta_{k'}^*\alpha_{k'}\rangle. \qquad (10.19)$$

The average $\langle\beta_k^*\alpha_k\rangle$, we may note from Eqs. (5.11) and (6.5), is just the mean number of quanta in the kth mode,

$$\langle\beta_k^*\alpha_k\rangle=\mathrm{tr}\{\rho a^\dagger{}_k a_k\}=\langle n_k\rangle. \qquad (10.20)$$

We have thus shown that the general expression (10.10) for the first-order correlation function always satisfies Eq. (10.14) when the field is described by a stationary density operator. The derivation of the equations relating the energy spectrum to the time-dependent correlation function then proceeds as before.

The simplest and most universal example of an incoherent field is the type generated by superposing the outputs of stationary sources. We have shown in some detail in Sec. VIII that as the number of sources which contribute to the excitation of a single mode increases, the density operator for the mode takes on a Gaussian form in the P representation. It is not difficult to derive an analogous result for the case of sources which excite many modes at once. We shall suppose that the sources ($j=1\cdots N$) are essentially identical, and that their contributions to the excitation are described by a weight function $p(\{\alpha_{jk}\})$. The weight function $P(\{\alpha_k\})$ for the superposed fields is then given by the convolution theorem as

$$P(\{\alpha_k\})=\int\prod_k\delta^{(2)}\Big(\alpha_k-\sum_{j=1}^{N}\alpha_{jk}\Big)\prod_{j=1}^{N}p(\{\alpha_{jk}\})\prod_k d^2\alpha_{jk}. \qquad (10.21)$$

Since the individual sources are assumed to be stationary, the function $p(\{\alpha_{jk}\})$ will only depend on the variables α_{jk} through their absolute magnitudes, $|\alpha_{jk}|$.

The derivation which leads from Eq. (10.21) to a Gaussian asymptotic form for $P(\{\alpha_k\})$ is so closely parallel to that of Eqs. (8.1)–(8.8) that there is no need to write it out in detail. The argument makes use of second-order moments of the function p which may, with the same type of vector notation used previously, be written as

$$\langle\boldsymbol{\alpha}_k\boldsymbol{\alpha}_{k'}\rangle=\int\boldsymbol{\alpha}_k\boldsymbol{\alpha}_{k'}p(\{\boldsymbol{\alpha}_k\})\prod_l d^2\alpha_l. \qquad (10.22)$$

The stationary character of the function p implies that such moments vanish for $k\neq k'$. With this observation, we may retrace our earlier steps to show that the many-dimensional Fourier transform of P takes the form of a product of Gaussians, one for each mode and each similar in form to that of Eq. (8.6). It then follows immediately that the weight function P for the field as a whole is given by a product of Gaussian factors each of the form of Eq. (8.8). We thus have

$$P(\{\alpha_k\})=\prod_k\frac{1}{\pi\langle n_k\rangle}e^{-|\alpha_k|^2/\langle n_k\rangle}, \qquad (10.23)$$

where $\langle n_k\rangle$ is the average number of photons present in the kth mode when the fields are fully superposed. One of the striking features of this weight function is its factorized form. It is interesting to remember, therefore, that no assumption of factorizability has been made regarding the weight functions p which describe the individual sources. These sources may, indeed, be ones for which the various mode amplitudes are strongly coupled in magnitude. It is the stationary property of the sources which leads, because of the vanishing of the moments (10.22) for $k\neq k'$, to the factorized form for the weight function (10.23).

The density operator which corresponds to the Gaussian weight function (10.23) evidently describes an ideally random sort of excitation of the field modes. We may reasonably surmise that it applies, at least as a good approximation, to all of the familiar sorts of incoherent sources in laboratory use. It is clear, in particular, from the arguments of Sec. VII that the Gaussian weight function describes thermal sources

[21] The Wiener-Khintchine theorem is usually expressed in terms of cosine transforms since it deals with a real-valued correlation function for the classical field $\mathbf{E}$, rather than a complex one for the fields $\mathbf{E}^{(\pm)}$. The complex correlation functions are considerably more convenient to use for quantum mechanical purposes, as is shown in Ref. 3.

correctly. The substitution of the Planck distribution $\langle n_k\rangle=[\exp(\hbar\omega_k/\kappa T)-1]^{-1}$ into Eq. (10.23) leads to the density operator for the entire thermal radiation field. To the extent that the Gaussian weight function (10.23) may describe radiation by a great variety of incoherent sources there will be certain deep-seated similarities in the photon fields generated by all of them. One may, for example, think of these sources all as resembling thermal ones and differing from them only in the spectral distributions of their outputs. As a way of illustrating these similarities we might imagine passing blackbody radiation through a filter which is designed to give the spectral distribution of the emerging light a particular line profile. We may choose this artificial line profile to be the same as that of some true emission line radiated, say, by a discharge tube. We then ask whether measurements carried out upon the photon field can distinguish the true emission-line source from the artificial one. If the radiation by the discharge tube is described, as we presume, by a Gaussian weight function, it is clear that the two sources will be indistinguishable from the standpoint of any photon counting experiments. They are equivalent sorts of narrow-band, quantum-mechanical noise generators.

It is a simple matter to find the correlation functions for the incoherent fields[2] described by the Gaussian weight function (10.23). If we substitute this weight function into the expansion (10.9) for the first-order correlation function we find

$$G_{\mu\nu}^{(1)}(\mathbf{r}t,\mathbf{r}'t')=\tfrac{1}{2}\sum_k \hbar\omega u_{k\mu}^*(\mathbf{r})u_{k\nu}(\mathbf{r}')\langle n_k\rangle e^{i\omega(t-t')}. \quad (10.24)$$

When the mode functions $\mathbf{u}_k(\mathbf{r})$ are the plane waves of Eq. (2.9), and the volume of the system is sufficiently large, we may write the correlation function as the integral

$$G_{\mu\nu}^{(1)}(\mathbf{r}t,\mathbf{r}'t')=\frac{\hbar c}{2(2\pi)^3}\int\sum_\lambda e_\mu^{(\lambda)*}e_\nu^{(\lambda)}\langle n_{\mathbf{k},\lambda}\rangle k \times\exp\{-i[\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')-\omega(t-t')]\}d\mathbf{k}, \quad (10.25)$$

in which the index λ again labels polarizations. To find the second-order correlation function defined by Eq. (10.2) we may write it likewise as an expansion in terms of mode functions. The only new moments of the weight function which we need to know are those given by $\langle|\alpha_k|^4\rangle=2\langle|\alpha_k|^2\rangle^2=2\langle n_k\rangle^2$. We then find that the second-order correlation function may be expressed in terms of the first-order function as

$$G_{\mu_1\mu_2\mu_3\mu_4}^{(2)}(x_1x_2,x_3x_4)=G_{\mu_1\mu_3}^{(1)}(x_1,x_3)G_{\mu_2\mu_4}^{(1)}(x_2,x_4) +G_{\mu_1\mu_4}^{(1)}(x_1,x_4)G_{\mu_2\mu_3}^{(1)}(x_2,x_3). \quad (10.26)$$

It is easily shown that all of the higher order correlation functions as well reduce to sums of products of the first-order function. The nth-order correlation function may be written as

$$G_{\mu_1\cdots\mu_{2n}}^{(n)}(x_1\cdots x_n,x_{n+1}\cdots x_{2n})=\sum_{\mathcal{P}}\prod_{j=1}^{n}G_{\mu_j\nu_j}^{(1)}(x_j,y_j), \quad (10.27)$$

where the indices ν_j and the coordinates y_j for $j=1\cdots n$ are a permutation of the two sets $\mu_{n+1}\cdots\mu_{2n}$ and $x_{n+1}\cdots x_{2n}$, and the sum is carried out over all of the $n!$ permutations. One of the family resemblances which links all fields represented by the weight function (10.23) is that their properties may be fully described through knowledge of the first-order correlation function.

The fields which have traditionally been called coherent ones in optical terminology are easily described in terms of the first-order correlation function given by Eq. (10.25). Since the light in such fields is accurately collimated and nearly monochromatic, the mean occupation number $\langle n_{\mathbf{k},\lambda}\rangle$ vanishes outside a small volume of $\mathbf{k}$-space. The criterion for accurate coherence is ordinarily that the dimensions of this volume be extremely small in comparison to the magnitude of $\mathbf{k}$. It is easily verified, if the field is fully polarized, and the two points $(\mathbf{r},t)$ and $(\mathbf{r}',t')$ are not too distantly separated, that the correlation function (10.25) falls approximately into the factorized form of Eq. (2.4). That is to say, fields of the type we have described approximately fulfill the condition for first-order coherence.[3] It is easily seen, however, from the structure of the higher order correlation functions that these fields can never have second or higher order coherence. In fact, if we evaluate the function $G^{(n)}$ given by Eq. (10.27) for the particular case in which all of the coordinates are set equal, $x_1=\cdots=x_{2n}=x$, and all of the indices as well, $\mu_1=\cdots=\mu_{2n}=\mu$, we find the result

$$G_{\mu\ldots\mu}^{(n)}(x\cdots x,x\cdots x)=n![G_{\mu\mu}^{(1)}(x,x)]^n. \quad (10.28)$$

The presence of the coefficient $n!$ in this expression is incompatible with the factorization condition (10.4) for the correlation functions of order n greater than one. The absence of second or higher order coherence is thus a general feature of stationary fields described by the Gaussian weight function (10.23). There exists, in other words, a fundamental sense in which these fields remain incoherent no matter how monochromatic or accurately collimated they are. We need hardly add that other types of fields such as those generated by radio transmitters or masers may possess arbitrarily high orders of coherence.

During the completion of the present paper a note by Sudarshan[12] has appeared which deals with some of the problems of photon statistics that have been treated here.[22] Sudarshan has observed the existence of what

[22] In an accompanying note, L. Mandel and E. Wolf [Phys. Rev. Letters **10**, 276 (1963)] warmly defend the classical approach to photon problems. Some of the possibilities and fundamental limitations of this approach should be evident from our earlier work. We may mention that the "implication" they draw from Ref. 1 and disagree with cannot be validly inferred from any reading of that paper.

we have called the P representation of the density operator and has stated its connection with the representation based on the n-quantum states. To that extent, his work agrees with ours in Secs. VII and IX. He has, however, made a number of statements which appear to attach an altogether different interpretation to the P representation. In particular, he regards its existence as demonstrating the "complete equivalence" of the classical and quantum mechanical approaches to photon statistics. He states further that there is a "one-to-one correspondence" between the weight functions P and the probability distributions for the field amplitudes of classical theory.

The relation between the P representation and classical theory has already been discussed at some length in Secs. VII–IX. We have shown there that the weight function $P(\alpha)$ is, in general, an intrinsically quantum-mechanical structure and not derivable from classical arguments. In the limit $\hbar \to 0$, which corresponds to large amplitudes of excitation for the modes, the weight functions $P(\alpha)$ may approach classical probability functions as asymptotic forms. Since infinitely many quantum states of the field may approach the same asymptotic form, it is clear that the correspondence between the weight functions $P(\alpha)$ and classical probability distributions is not at all one-to-one.

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